THE SCHWARZIAN DERIVATIVE IN RIEMANNIAN GEOMETRY, UNIVALENCE CRITERIA AND QUASICONFORMAL REFLECTIONS

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By
Martin S. Chuaqui
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Brad G. Osgood
(Principal Adviser)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Halsey L. Royden

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Rafe Mazzeo

Approved for the University Committee on Graduate Studies:

Dean of Graduate Studies & Research

Abstract

In classical complex analysis, the Schwarzian derivative has played a key role as a means of characterizing sufficient conditions for the univalence of a locally injective analytic map.

Osgood and Stowe have recently introduced a notion of Schwarzian derivative for conformal local diffeomorphisms of Riemannian manifolds which generalizes the classical operator in the plane. Using their new definition, they derive a sufficient condition for a conformal local diffeomorphism ψ of a Riemannian n-manifold (M, g) to the standard sphere (S^n, g_1) to be injective (O-S). By setting $M = D$ the unit disc in the plane and g alternately the euclidean and the hyperbolic metric, Osgood and Stowe obtain from their result two classical criteria of Nehari.

During the past two years I have studied O-S and its implications. By considering other kind of metrics in D, I derive from O-S most of the known and some new univalence criteria that involve either the Schwarzian derivative of ψ or the quantity ψ''/ψ'. In particular, a recent criterion of Epstein can be obtained in this fashion.

It is often the case that a stronger form of a given univalence criterion serves further as a condition that guarantees a quasiconformal extension to the entire plane. With the aid of Epstein's techniques for constructing reflections in hyperbolic $n+1$-space, we show that indeed a strong form of O-S implies the existence of a quasiconformal reflection on S^n, which fixes pointwise the boundary of the image $\psi(M)$. We follow Ahlfors in his definition of quasiconformality in higher dimensions. The main point in proving this theorem, is that the quasiconformal distortion of the reflection which is determined by the support function ρ defined on $\psi(M)$ by $e^{2\rho} g_1 = (\psi^{-1})^*(g)$ can be expressed naturally in terms of the quantities in O-S.

An unexpected phenomenon is that the existence of a map ψ satisfying the strong form of O-S implies that M is simply-connected. This has the interesting consequence that certain types of univalence criteria (which do hold on simply-connected domains) cannot exist on domains of higher connectivity. By means of conformal invariance, we restate the strong form of O-S as a (sharp) sufficient condition for a domain on the S^n to be simply-connected.

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Contents

Chapter 1

Introduction

1.1 Motivation of the problem

In classical complex analysis, the Schwarzian derivative has played a key role as a means of characterizing necessary and sufficient conditions for the univalence of a locally injective analytic map. In the unit disc, there is mainly one necessary condition for global univalence, proved originally by Kraus in 1932 [32] and commonly attributed to Nehari, who discovered it independently in 1949 [36]. His proof uses the so called "area theorem", or equivalently, coefficient estimates for the power series expansion in the disc of the given analytic map. Another proof of this result was given by Bergman and Schiffer using the theory of kernel functions and conformal mappings [14]. Such an approach had the advantage that it could be applied to obtain similar results for other simply- and even multiply-connected domains. On the other hand, for the sufficiency of univalence of analytic maps defined say in the unit disc, many apparently different criteria have been established. Since in many cases their proofs have relied on similar arguments, there has been an interest in deriving general criteria which comprised as many as possible of the known results. Also, frequently a stronger form of a given injectivity criterion can serve further as a sufficient condition for the existence of quasiconformal extensions to the entire plane. As a main step in understanding the phenomenon of injectivity and eventually quasiconformal extension, Epstein has proved recently a remarkable theorem which generalizes many such known results

[22]. His approach is mainly differential geometric and uses in a beautiful way the geometry of hyperbolic 3-space. In quite a different character and in a way, with a more classical approach, Anderson and Hinkkanen established also recently an even stronger sufficient criterion for univalence and quasiconformal extension [9]. Their theorem is more general, in that it applies to analytic maps defined on quasidiscs.

A generalization of the notion of Schwarzian derivative to higher dimensions was considered by Ahlfors in [6]. He discusses that concept for local diffeomorphisms in $R^n, n \geq 3$, by making an analogy with the real and imaginary parts of the usual Schwarzian derivative of analytic maps in the plane. In euclidean space of dimension ≥ 3, all conformal maps are Möbius transformations and as in the case $n = 2$, their Schwarzian derivative vanishes identically. So in some sense, the conformal significance of the Schwarzian derivative in R^n is trivial for $n \geq 3$. In their paper "The Schwarzian derivative and conformal mappings of Riemannian manifolds" [41], Osgood and Stowe introduce a notion of Schwarzian derivative on manifolds, which generalizes the classical operator in the plane. In the subsequennt paper "A generalization of Nehari's univalence criterion" [42], and using their new notion, these authors establish a sufficient condition for the injectivity of a conformal local diffeomorphism of an n-dimensional Riemannian manifold M to the standard sphere S^n. The idea of their proof, which can be partially traced back to some classical proofs, is to translate the given inequality on the Schwarzian to a differential inequality along geodesics. Then they apply a standard Sturm comparison theorem for ordinary differential equations. They obtain as corollaries, with M the unit disc in the plane and particular choices of its metric, two classical criteria of Nehari.

In Chapter 2, we define cross-ratio on Riemannian manifolds, which we shall show relates to the Schwarzian derivative of Osgood and Stowe in a way analogous to the relationship between the two quantities in the plane. In particular, the Schwarzian derivative can be viewed as the first nontrivial term in the infinitesimal deformation of cross-ratio.

Chapter 3 will be devoted to deriving from the general theorem of Osgood and Stowe some new and most of the known injectivity criteria. In particular, we shall obtain in this fashion the injectivity result of Epstein. The language of conformal geometry in which the result of Osgood and Stowe is stated, allows us to obtain a sufficient condition for the univalence of analytic maps defined on arbitrary simply-connected domains. This theorem can be considered as the counterpart to the necessary condition established by Bergman and Schiffer. It comes as somewhat of a surprise to realize that an equivalent sufficient condition cannot exist on domains of multiple connectivity. We shall prove a general theorem to that extent. By means of conformal invariance, we can restate this result as a sufficient condition for a domain on the S^n to be simply-connected. The theorem is sharp.

In Chapter 4, the differential geometric techniques of Epstein in hyperbolic $n+1$-space will be used to show when a strengthened version of the univalence criterion of Osgood and Stowe guarantees the existence of a quasiconformal reflection on the target space S^n. The key point will be the fact that the distortion of the reflection across a hypersurface in H^{n+1}, which is determined by a support function ρ on a domain in $S^n = \partial H^{n+1}$, can be expressed in terms of the main quantities involved in the theorem of Osgood and Stowe (namely, the Schwarzian derivative of ρ and the scalar curvature of the metric $e^{2\rho}g_1$, where g_1 is the round metric on the sphere).

In Chapter 5, we will investigate some variations of the theorem of Osgood and Stowe. In short, a very important ingredient in the proof of their theorem is the existence of good "test functions" on S^n, i.e., functions $u \geq 0$ vanishing only at a given point, for which $u^{-2}g_1$ is flat and such that the Hessian of u (in the spherical metric) is proportional to g_1 itself. The same holds true in R^n and hyperbolic space H^n with their respective metrics of constant curvature 0 and -1. In R^n, such functions are given by the square of the distance to a given point. On the other hand, functions u for which $Hess(u) - \frac{1}{n}(\Delta u)g$ is small always exist locally on any Riemannian manifold with metric g; simply take $u = dist^2(\ ,P_0)$ in a neighborhood of the point P_0. In fact, the norm of $Hess(u) - \frac{1}{n}(\Delta u)g$ is $O(u)$ near P_0. In the case when the target manifold is complete, simply-connected and of nonpositive curvature, say bounded between $-a^2$ and 0, the function $u = dist^2(\ ,P_0)$ is smooth everywhere. It is possible to appropriately estimate the quantity $Hess(u) - \frac{1}{n}(\Delta u)g$ by using comparison theorems. This will yield an injectivity criterion for conformal maps of an arbitrary Riemannian manifold into such target spaces.

The other situation in which one can obtain nonnegative choices of u vanishing at only one point and for which $Hess(u) - \frac{1}{n}(\Delta u)$ is well behaved, is when considering the restriction of solutions to $Hess(u) = \frac{1}{m}(\Delta u)g_0$ in R^m, $m > n$, to an n-dimensional submanifold with the induced metric. In that case, we shall obtain an injectivity criterion for a conformal immersion of a manifold into higher dimensional euclidean space. By means of two examples we will show this theorem to be sharp. The criterion is much simpler when the immersion is isometric, and since holomorphic maps on domains in C^n or for that matter, any complex manifold, are conformal on complex lines, the theorem also has particular formulations in the complex analytic setting.

1.2 Some background

In this section we will set up basic notation, give a few definitions and establish some preliminary results.

Let M be an n-dimensional Riemannian manifold with metric g. When $M = R^n$, we will denote by g_0 the euclidean metric and on the n-sphere S^n, g_1 will stand for the standard round metric. Given a conformal metric $\hat{g} = e^{2\varphi}g$ on M, Osgood and Stowe define the Schwarzian tensor of $\hat{g}$ with respect to g as the symmetric, traceless (0,2)-tensor

$$B_g(\varphi) = Hess(\varphi) - d\varphi \otimes d\varphi - \frac{1}{n}(\Delta\varphi - |grad\ \varphi|^2)g\,, \tag{1.2.1}$$

where the metric dependent quantities on the right-hand side are computed with respect to the metric g. When ψ is a conformal local diffeomorphism of (M, g) to another Riemannian manifold (N, g'), then $\psi^*(g') = e^{2\varphi}g$ with $\varphi = \log|D\psi|$. The Schwarzian derivative of ψ is defined by

$$S_g(\psi) = B_g(\varphi)\,. \tag{1.2.2}$$

For an analytic map ψ in the plane, with $g = g' = g_0$, then $\varphi = \log|\psi'|$ and computing in standard coordinates one gets

$$S_g(\psi) = \begin{pmatrix} Re\{\psi, z\} & -Im\{\psi, z\} \\ -Im\{\psi, z\} & -Re\{\psi, z\} \end{pmatrix}\,, \tag{1.2.3}$$

where $\{\psi, z\} = (\frac{\psi''}{\psi'})' - \frac{1}{2}(\frac{\psi''}{\psi'})^2$ is the classical Schwarzian derivative.

On M, the conformal metric $\hat{g} = e^{2\varphi}g$ is called Möbius with respect to g if $B_g(\varphi) = 0$, and so a conformal local diffeomorphism ψ as before is said to be Möbius if $S_g(\psi) = 0$. If φ and σ are smooth functions on M, then there is an important identity:

$$B_g(\varphi + \sigma) = B_g(\varphi) + B_{\hat{g}}(\sigma)\,, \tag{1.2.4}$$

where $\hat{g} = e^{2\varphi}g$. In a chain of conformal local diffeomorphisms $\psi_1 : (M, g) \to (N_1, g')$ and $\psi_2 : (N_1, g') \to (N_2, g'')$, equation (1.2.4) can be formulated as

$$S_g(\psi_2 \circ \psi_1) = S_g(\psi_1) + \psi_1^*(S_{g'}(\psi_2))\,. \tag{1.2.5}$$

This reduces to the classical formula for the Schwarzian derivative of a composition of analytic maps in the plane. The other important fact, also analogous to the situation in the complex plane, is that the nonlinear equation

$$B_g(\varphi) = 0 \tag{1.2.6}$$

transforms under substitution $u = e^{-\varphi}$ to the linear equation

$$Hess(u) = \frac{1}{n}(\Delta u)g\,, \tag{1.2.7}$$

and more generally

$$B_g(\varphi) = q \tag{1.2.8}$$

linearizes under the change above to

$$Hess(u) - \frac{1}{n}(\Delta u)g = -uq\,. \tag{1.2.9}$$

For proofs and further references on the preceedings we refer the reader to [41].

For conformal local diffeomorphisms ψ of R^n to R^n, the vanishing of $S_{g_0}(\psi)$ coincides with the classical definition of Möbius transformations (defined as composites of dilations, rotations, translations and inversions). The difference between the planar case and the case when $n \geq 3$ is that in the latter, Möbius transformations are the only (even locally defined) conformal maps. This is the well-known theorem of

Liouville. If we denote by conf(M, g) and Möb(M, g) respectively the groups of conformal and Möbius diffeomorphisms of M to itself, then more generally, for $n \geq 3$, conf$(M, g) =$ Möb(M, g) when (M, g) is Einstein (see [41]).

On arbitrary surfaces, the following theorem in [41] characterizes Möbius transformations

Theorem 1.2.1 *A conformal diffeomorphism between surfaces is a Möbius transformation iff it maps (all) curves of constant geodesic curvature to curves of constant geodesic curvature.*

Proof: Let $\psi : (M, g) \to (M', g')$ be a conformal diffeomorphism between surfaces, and let $\psi^*(g') = e^{2\varphi} g$. If γ is a curve in M with unit tangent T, then its geodesic curvature is defined by

$$\nabla_T T = kN\,,$$

where ∇ is the covariant derivative in (M, g) and N a unit normal to γ. Using the change of the covariant derivative under conformal change of metric

$$\widehat{\nabla}_X Y = \nabla_X Y + (X\varphi)Y + (Y\varphi)X - g(X, Y) grad\ \varphi\,, \tag{1.2.10}$$

one finds that the geodesic curvature of γ in $\hat{g} = e^{2\varphi} g$ is given by

$$\hat{k} = e^{-\varphi}(k - (N\varphi))\,. \tag{1.2.11}$$

Differentiating (1.2.11) with respect to T yields

$$e^{2\varphi}(\hat{T}\hat{k}) = (Tk) - B_g(\varphi)(T, N)\,, \tag{1.2.12}$$

with $\hat{T} = e^{-\varphi} T$. The theorem now follows from the fact that since $B_g(\varphi)$ is symmetric and traceless, it is determined by its action on pairs of orthogonal tangent vectors.

For 3-manifolds we establish the following

Theorem 1.2.2 *A conformal diffeomorphism between 3-dimensional manifolds is Möbius iff it maps (all) curves of constant geodesic curvature and zero torsion to curves of constant geodesic curvature and zero torsion.*

Proof: Let $\psi : (M, g) \to (M', g')$ be a conformal diffeomorphism between 3-manifolds, and let $\psi^*(g') = e^{2\varphi}g$. For a curve γ in M with unit tangent T we have

$$\nabla_T T = kN \quad \text{and} \quad \nabla_T N = \tau B - kT\,,$$

where N and B are respectively unit normal and binormal to γ, k is the geodesic curvature and τ the geodesic torsion of γ. Using (1.2.10), one now verifies that

$$e^{2\varphi}\hat{k}\hat{N} = kN + (T\varphi)T - grad\ \varphi\,.$$

Note that we no longer have necessarily $\hat{N} = e^{-\varphi}N$. In any case, $g(\hat{N}, T) = 0$, $g(\hat{N}, N) = k - (N\varphi)$ and $g(\hat{N}, B) = B\varphi$. By taking norms we obtain

$$e^{2\varphi}\hat{k}^2 = (k - (N\varphi))^2 + (B\varphi)^2\,. \tag{1.2.13}$$

After differentiating this equation, a short calculation leads to

$$\begin{aligned} e^{3\varphi}\hat{k}(\hat{T}\hat{k}) \quad &= \quad (k - (N\varphi))((Tk) - B_g(\varphi)(T, N)) - \\ &\quad\ (B\varphi)(k\tau - B_g(\varphi)(T, B))\,. \end{aligned} \tag{1.2.14}$$

Since $\hat{\nabla}_{\hat{T}}\hat{N} = \hat{\tau}\hat{B} - \hat{k}\hat{T}$, we conclude that

$$\hat{k}\hat{N} = e^{-2\varphi}((k - N\varphi)N - (B\varphi)B)\,.$$

But $\nabla_T B = -\tau N$, and now a long yet elementaty calculation yields

$$e^{\varphi}\hat{\tau} = \frac{(B\varphi)(Tk - B_g(\varphi)(T, N)) + (k - N\varphi)(k\tau - B_g(\varphi)(T, B))}{(k - N\varphi)^2 + (B\varphi)^2}\,. \tag{1.2.15}$$

Suppose now that $B_g(\varphi) = 0$. Then $Tk = \tau = 0$ implies $\hat{T}\hat{k} = \hat{\tau} = 0$. On the other hand, assume that under the conformal change of metric, curves mantain the property of having constant geodesic curvature and zero torsion. Given a point $p \in M$ where $grad\ \varphi \neq 0$ and two orthogonal vectors $T, N \in T_pM$, let γ be the geodesic with tangent vector T at p. Thus $k = 0$. Let $B \in T_pM$ be orthogonal to T and N, and let T, N, B stand also for the parallel translates of these vectors along γ.

Suppose that *grad* φ is not a multiple of T at that point. Hence $(N\varphi)^2+(B\varphi)^2 \neq 0$ near p and from (1.2.14) and (1.2.15) we get

$$-(N\varphi)B_g(\varphi)(T,N) + (B\varphi)B_g(\varphi)(T,B) = 0$$

and

$$-(B\varphi)B_g(\varphi)(T,N) - (N\varphi)B_g(\varphi)(T,B) = 0\,,$$

hence

$$B_g(\varphi)(T,N) = B_g(\varphi)(T,B) = 0\,.$$

Now by continuity in the arguments T and N, we also have that $B_g(\varphi)(T,N) = 0$ when $N\varphi = B\varphi = 0$. This shows that $B_g(\varphi) = 0$ at such p. In the interior of the set where *grad* $\varphi = 0$, by its definition, $B_g(\varphi) = 0$ and its vanishing everywhere follows again by continuity.

This characterization of Möbius transformations allows the following inductive proof of Liouville's theorem, which states

Theorem 1.2.3 *Let $\Omega \subset R^n, n \geq 3$, be an open and connected set, and let $\psi : \Omega \to R^n$ be a conformal local diffeomorphism. Then ψ is the restriction to Ω of a Möbius transformation of $R^n \cup \{\infty\}$.*

For the proof, we may assume by taking if necessary a subset of Ω, that ψ is injective. Also, just to fix ideas, by composing ψ with a Möbius transformation, we may also assume that $0 \in \Omega$ and that ψ fixes the origin. In R^n, the only totally umbilic hypersurfaces are (pieces) of hyperplanes and spheres [40]. Let Σ be a totally umbilic hypersurface, and as before, let $\psi^*(g_0) = e^{2\varphi}g_0$. Under this conformal change of metric one has

$$e^{\varphi}X\hat{H} = XH - B_g(\varphi)(X,N)\,,$$

where H and $\hat{H}$ are respectively the mean curvatures of Σ in g_0 and $e^{2\varphi}g_0$, X is tangent to Σ and N is the normal vector (see [41]). Since the curvatures of g_0 and $e^{2\varphi}g_0$ vanish and $n \geq 3$, we must have that $B_g(\varphi) = 0$. (This is basically the proof that all conformal selfmaps of Einstein manifolds are actually Möbius, and it follows from the fact that $B_g(\varphi)$ is the term by which traceless part of the Ricci tensor changes under a conformal change of metric [41].)

Since the property of being totally umbilic is preserved under conformal maps, we conclude that ψ takes (pieces) of hyperplanes and spheres to (pieces) of hyperplanes and spheres.

We now use an induction argument. Let $n = 3$ and Σ_0 be a plane through the origin. Then $\psi(\Sigma_0)$ lies on a plane or a sphere. From the characterization of Möbius maps in 3-manifolds, we get that ψ takes circles (closed curves of constant geodesic curvature and zero torsion) into circles. Thus the restriction of ψ to $\Omega \cap \Sigma_0$ is a 2-dimensional Möbius map F_0, which can be extended uniquely to a Möbius map of $R^3 \cup \{\infty\}$. If now Σ is any plane through the origin which is orthogonal to Σ_0, the same argument shows that the restriction of ψ to $\Omega \cap \Sigma$ is Möbius. But since these two conceivably different Möbius transformations agree on $\Omega \cap \Sigma_0 \cap \Sigma$, they have to agree everywhere.

Assuming that the result is valid for $n - 1$, we will show it for n in a similar fashion.

Let Σ_0 be a hyperplane through the origin. The induction hypothesis gives that the restriction of ψ to $\Omega \cap \Sigma_0$ is a Möbius map T_0 in $n-1$ dimensions, which again can be extended uniquely to a Möbius map in n dimensions. If Σ is any hyperplane through the origin containing the direction normal to Σ_0, then as before, the restriction of ψ to such hyperplane is Möbius and agrees with T_0 on $\Omega \cap \Sigma_0 \cap \Sigma$. Hence, they have to be equal everywhere. This finishes the proof.

Another way of understanding conformal transformations in euclidean space is by studying the associated Lie algebra, i.e., vector fields that generate a 1-parameter family of diffeomorphisms which are conformal. Ahlfors has introduced the following operator as a way of generalizing the operator $\overline{\partial}$: given $V : \Omega \subset R^n \to R^n$ differentiable, he defines

$$SV(x) = \frac{1}{2}(DV(x) + DV^t(x)) - \frac{1}{n}(trDV(x))I\,, \tag{1.2.16}$$

where Ω is an open set, $DV(x)$ the differential of V at x, $DV^t(x)$ its transpose and I the identity matrix. It is not difficult to see that when $n = 2$, $SV = 0$ iff V is conformal. Ahlfors calls the solutions to $SV = 0$ trivial deformations and shows that for $n \geq 3$, the only solutions are of the form

$$V(x) = a + B(x) + 2(c \cdot x)x - |x|^2 c\,, \tag{1.2.17}$$

where $a, c \in R^n$, $\cdot$ is the euclidean inner product, $|x|$ the euclidean length, and B is a linear map satisfying $SB = 0$ [4].

Sarvas established a close connection between Ahlfors' characterization of trivial deformations and the theorem of Liouville. In order to relate both theorems, he analyzes the flow generated by V, i.e., the solution to the differential equation

$$\frac{\partial}{\partial t}\psi(x,t) = V(\psi(x,t)) \ \ , \ \psi(x,0) = x\,, \tag{1.2.18}$$

and proves the following key result:

Lemma 1.2.1 *The solution to (1.2.18) consists of conformal maps $\psi_t(x) = \psi(x,t)$ for all sufficiently small t iff V is a trivial deformation.*

We refer the reader to [49] for the proof. Using this, he shows that for $n \geq 3$, Ahlfors' characterization of trivial deformations can be rather easily derived from Liouville's theorem, and vice-versa.

On the other hand, in the general setting of Riemannian geometry it is known that a vector field V generates a conformal flow iff its covariant differential satisfies

$$\nabla_X V = vX + \sigma(X)\,, \tag{1.2.19}$$

where v is a function and σ is a field of skew-symmetric linear endomorphisms. Furthermore, the flow consists of isometries iff v is zero and of Möbius transformations iff $Hess(v) = \frac{1}{n}\Delta(v)g$. In R^n, the solutions to this last equation are given by

$$v(x) = \mathrm{a}|x|^2 + b \cdot x + \mathrm{c}\,, \tag{1.2.20}$$

where a, c $\in R$, $b \in R^n$ [41]. Therefore, Liouville's theorem implies that for $n \geq 3$, all vector fields $V : \Omega \subset R^n \to R^n$ generating a conformal flow satisfy (1.2.19), with v as in (1.2.20).

Let us denote by G the group of all Möbius transformations in R^n, and let TG be its tangent space at the identity. We can tie together the various notions discussed above, and show

Proposition 1.2.1 *The following are equivalent:*
(A) V *generates a conformal flow*,
(B) $\nabla_X V = vX + \sigma(X)$, *as in (1.2.19)*,
(C) V *is as in (1.2.17)*,
(D) $SV = 0$,
(E) $V \in TG$.

We will understand the last condition as saying that for some $\epsilon > 0$, there exist a curve $\gamma : (-\epsilon, \epsilon) \to G$ with $\gamma(0) = I$ and $\gamma'(0) = V$.

Proof: The equivalence of (A) and (B) has already been established, and (C) iff (D) is Ahlfors' theorem. Sarvas' lemma is precisely (A) iff (D), but we want to show this in a slightly different way. It is easy to see that (C) implies (B), and we will show that (B) implies (C). Then we will prove that (C) iff (E).

So, assume that V satisfies B. In the proof, we will require V to be at least C^4, but as Sarvas shows in his paper, weaker regularity conditions are sufficient. We may also asssume that the domain Ω contains the origin and that $V(0) = 0$. Let us write $V = (V_1, ..., V_n)$ and $X_i = \partial_i$ for coordinate differentiation. At $x \in \Omega$, $\sigma = (\sigma_{ij})$ is skew-symmetric. Then,

$$< \nabla_{X_j} V, X_i >= \partial_j V_i = v\delta^i_j + \sigma_{ij} \,,$$

where we have used $< \,,\, >$ to denote the euclidean inner product. Hence,

$$\partial_j V_i = \begin{cases} v & \text{if } i = j \\ \sigma_{ij} & \text{if } i \neq j \end{cases} .$$

Recall that $v(x) = \mathrm{a}|x|^2 + b \cdot x + \mathrm{c}$, so that

$$\partial_2^2 \partial_1 V_1 = \partial_2^2 v = 2\mathrm{a} = \partial_1 \partial_2 \sigma_{12} = -\partial_1 \partial_2 \sigma_{21} = -\partial_1^2 \partial_2 V_2 = -\partial_1^2 v = -2\mathrm{a} \,,$$

thus a$= 0$. We will show now that for $i \neq j$

$$\partial_j V_i = b_j x_i - b_i x_j + \sum_{k \neq i,j} a^{ij}_k x_k + c_{ij} \,,$$

where a_k^{ij} are functions satisfying $a_k^{ij} = a_j^{ik} = -a_k^{ji}$, and c_{ij} constants such that $c_{ij} = -c_{ji}$. Here, $b = (b_1, ..., b_n)$. We know that for $i \neq j$, $\partial_j V_i = \sigma_{ij}$, hence

$$\partial_i \sigma_{ij} = \partial_i \partial_j V_i = \partial_j v = b_j \,,$$

and

$$\partial_j \sigma_{ij} = -\partial_j \sigma_{ji} = -b_i \,.$$

Thus

$$\sigma_{ij} = b_j x_i - b_i x_j + \mu_{ij} \,,$$

where

$$\partial_i \mu_{ij} = \partial_j \mu_{ji} = 0 \,.$$

Let $k \neq i, j$. Then

$$\partial_k \mu_{ij} = \partial_k \sigma_{ij} = \partial_j \partial_k V_i = \partial_j \sigma_{ik} = \partial_j \mu_{ik} \,,$$

hence

$$\partial_k^2 \mu_{ij} = \partial_j \partial_k \mu_{ik} = 0 \,,$$

since μ_{ik} is independent of x_k. Therefore

$$\mu_{ij} = \sum_{k \neq i,j} a_k^{ij} x_k + c_{ij} \,.$$

The skew-symmetry of μ_{ij} in i and j follows from the one of σ_{ij}, and

$$a_k^{ij} = \partial_k \partial_j V_i = \partial_j \partial_k V_i = a_j^{ik} \,.$$

All the symmetries of the a_k^{ij} will force them to vanish, as

$$a_k^{ij} = a_j^{ik} = -a_j^{ki} = -a_i^{kj} = a_i^{jk} = a_k^{ji} = -a_k^{ij} \,.$$

Therefore we finally have

$$\partial_j V_i = \begin{cases} b \cdot x \; + c \,, & \text{if } i = j \\ b_j x_i - b_i x_j + c_{ij} & \text{if } i \neq j \end{cases} \text{, and } \; c_{ij} = -c_{ji} \,.$$

Let C be the skew-symmetric matrix (c_{ij}), and let $B = cI + C$. Then, $SB = 0$ (in fact, in general, $SB = 0$ iff $B = \lambda I + D$ with λ a constant, and $D^t = -D$). Let

$$W(x) = B(x) + (b \cdot x)x - \frac{1}{2}|x|^2 b \ .$$

It is easy to see that all derivatives of W and V are equal, and since $W(0) = V(0) = 0$, we conclude the desired equality.

We show now the implication (C)$\Rightarrow$(E). Let V be of the form

$$V(x) = a + B(x) + 2(c \cdot x)x - |x|^2 c \,,$$

where $a, c \in R^n,\ B = \lambda I + D,\ D^t = -D$. We identify each part of the decomposition, namely

$$a = \frac{dA_1}{dt}|_{t=0} \quad , \quad A_1(t) = I + ta,$$

$$B = \frac{dA_2}{dt}|_{t=0} \quad , \quad A_2(t) = e^{tB}$$

and

$$|x|^2 c - 2(c \cdot x)x = \frac{dA_3}{dt}|_{t=0} \,,$$

with $A_3(t) = J(J(x) + u(t))$. Here $J(x) = \frac{x}{|x|^2}$ and we may take $u(t)$ to be any curve such that $u(0) = 0$ and $u'(0) = c$.

To conclude that (E) implies (C) we count dimensions. A vector field V of the form (1.2.17) as in (C) has $n + (1 + \frac{n(n-1)}{2}) + n = \frac{(n+1)(n+2)}{2}$ degrees of freedom, which is precisely the dimension of the group G.

We make a final remark about the case $n = 2$. Liouville's theorem still holds if we talk about conformal diffeomorphisms of R^2, i.e., they are all Möbius transformations. Therefore, if $\Omega = R^2$ we again have (A) iff (B) iff (C) iff (E), and any of these conditions implies (D). But as mentioned before, (D) holds for any analytic function.

Chapter 2

Cross-Ratio

2.1 The Schwarzian derivative as an infinitesimal cross-ratio

In this section we will introduce a notion of cross-ratio on Riemannian manifolds, from which we will recover the Schwarzian tensor $B_g(\varphi)$ of the conformal metric $e^{2\varphi}g = \psi^*(g)$ as the first nontrivial term in the infinitesimal deformation of cross-ratio under the map ψ.

In his paper "Schwarzian derivatives and cross-ratios in R^n" [6], Ahlfors defines the cross-ratio of four points in R^n as follows: given $x, y, u, v, \in R^n$, there exists a unique 2-sphere (2-plane or line) passing through these four points. Thus you can view these points as complex numbers on a Riemann sphere, and their cross-ratio is therefore defined. Let us denote it by (x, y, u, v), which will be our notation for cross-ratio from now on. Ahlfors shows that its absolute value $|(x, y, u, v)|$ is given by

$$|(x, y, u, v)| = \frac{|x-u||y-v|}{|x-v||y-u|},$$

and the argument of (x, y, u, v) is the angle at u between circular arcs from u through x to v, and from u through y to v. This angle is unique up to change of sign, i.e., up to orientation of the given 2-sphere. With this definition, the absolute cross-ratio, meaning the absolute value of the cross ratio, becomes invariant under Möbius transformations.

The same idea can be used to define the cross-ratio of four points in any real n-dimensional vector space which is endowed with an inner product. It is now natural to consider the case of an n-dimensional Riemannian manifold (M, g), and study the cross-ratio of four vector fields, i.e., the cross-ratio will be taken at each point with the corresponding inner product coming from the metric g.

We will give now the formal definitions. Let X, Y, U, V be vector fields on M. The length of a vector will be denoted by $|\ |$. Then the cross-ratio (X, Y, U, V) is defined to be the complex number z, unique up to conjugation, satisfying

$$|z| = \frac{|X-U||Y-V|}{|X-V||Y-U|} \tag{2.1.1}$$

and

$$\cos(\arg z) = \frac{g(A,B)}{|A||B|}, \tag{2.1.2}$$

where A and B are given by

$$\begin{aligned} A &= \frac{X-V}{|X-V|^2} - \frac{U-V}{|U-V|^2}, \\ B &= \frac{Y-V}{|Y-V|^2} - \frac{U-V}{|U-V|^2}. \end{aligned} \tag{2.1.3}$$

This last expression defining the argument of the cross-ratio comes from the following: at a fixed point in M, we can perform a Möbius inversion about the point V, which will leave the cross-ratio invariant up to conjugation. The 2-sphere on the tangent space at that point gets mapped under the inversion above onto a plane through infinity on that tangent space. The circular arcs defining the argument of the cross-ratio are taken to straight lines, and now our definition follows from looking at the angle between these two lines. Note that whereas the argument of the cross-ratio is not unique, its cosine is well defined.

In the complex plane, there is an interesting and well-known relation between cross-ratio and Schwarzian derivative. It comes from then following identity: given ψ analytic at z and four complex numbers a, b, c, d, then

$$\begin{aligned} (\psi(z+ta), \psi(z+tb), \psi(z+tc), \psi(z+td)) &= (a,b,c,d)(1 + \frac{1}{6}(a-b)(c-d)\{\psi, z\}t^2 \\ &\quad + O(t^3)). \end{aligned} \tag{2.1.4}$$

A proof can be found in [6].

It is our purpose to generalize this last equation to the context of Riemannian manifolds, at least when considering the absolute cross-ratio.

Let $\psi : M \to M$ will be a conformal diffeomorphism. We write $\psi^*(g) = e^{2\varphi}g$. Given a point $p \in M$ and the cross-ratio in T_pM, we can define the cross-ratio of four points $p_1, p_2, p_3, p_4 \in M$ that are sufficiently close to p by using the exponential map at p, namely by

$$(p_1, p_2, p_3, p_4) = (exp_p^{-1}(p_1), exp_p^{-1}(p_2), exp_p^{-1}(p_3), exp_p^{-1}(p_4))\,.$$

Let now X_1, X_2, X_3, X_4 be four distinct vectors at p, and we seek an expression similar to the one that can be obtained from (2.1.4) for the absolute cross ratio

$$|(\psi(exp(tX_1)), \psi(exp(tX_2)), \psi(exp(tX_3)), \psi(exp(tX_4)))|\,.$$

For convenience, we have dropped the subindex in the exponential map. This will cause no confusion, since from the context one will know at which point it is based. For example, in the last equation the cross-ratio is at $q = \psi(p)$.

The main tool for deriving such an expansion will be to obtain the first few terms in the expansion in t of $exp^{-1}(\psi(exp(tX)))$, where $X \in T_pM$.

The conformal map ψ provides an isometry between the metrics g and $\hat{g} = e^{-2\sigma}g$, where $\sigma = \varphi \circ \psi^{-1}$. Because of this, ψ commutes with the corresponding exponentials, i.e., $\psi(exp(tX)) = \widehat{exp}(t\psi_*(X))$, where $\hat{}$ stands for quantities in the metric $\hat{g}$. Let γ the curve in T_qM given by $t \to exp^{-1} \circ \widehat{exp}(t\psi_*(X))$. We refer now to [41] for the following result:

Proposition 2.1.1 *A parametrization of γ by constant speed $|\psi_*(X)|$ in the metric $exp^*(g)$ is given by*

$$\begin{aligned}\mu(\tau) &= \tau Y - \frac{\tau^2}{2}(grad\ \sigma)^N|Y|^2 + \frac{\tau^3}{6}(\{M(-\sigma)Y\}^N - |\{grad\ \sigma\}^N|^2 Y) \\ &\quad + O(\tau^4|Y|^4)\,, \end{aligned} \tag{2.1.5}$$

where $Y = \psi_*(X)$, ^{N} stands for projection onto the orthogonal complement of the line $\mathrm{R}Y$, and $M(-\sigma)$ is the matrix representing the Schwarzian tensor of $B_g(-\sigma)$ in the *metric g, i.e.,*

$$B_g(-\sigma)(Y,Z) = g(M(-\sigma)Y,Z)\ ,$$

for all $Z \in T_pM$.

We shall use this result to prove the next

Proposition 2.1.2 *Let $\alpha(t) = exp^{-1}\circ \widehat{exp}(tY)$. Then*

$$\alpha(t) = tY + t^2A + t^3B + O(t^4)\,,$$

where

$$\begin{aligned} A &= X(\varphi)Y - \frac{1}{2}\psi_*(grad\ \varphi)\,, \\ 6B &= (2Hess(\varphi)(X,X) + 4X(\varphi)^2 - |X|^2|grad\ \varphi|^2)Y - \\ &\quad |X|^2\psi_*(\nabla_X grad\ \varphi) - 2X(\varphi)|X|^2\psi_*(grad\ \varphi)\,. \end{aligned} \tag{2.1.6}$$

Proof: Let $\beta(t) = \widehat{exp}(tY)$ and $\gamma(\tau) = exp(\mu(\tau))$. These represent the same curve in M, with $||\beta'(t)||$ constant equal to $||Y|| = e^{-\sigma}|Y|$ and $|\gamma'(\tau)|$ constant equal to $|Y|$ (here, we have denoted by $||\ ||$ the norm in the metric $\hat{g}$). We seek a change of parameter $t \to s(t)$ with $s(0) = 0$ such that the curve $\eta(t) = \widehat{exp}(s(t)Y)$ has constant speed $|Y|$ in the metric g. For then, $\eta(t) = \gamma(t)$. We compute:

$$\eta'(t) = D\widehat{exp}|_{s(t)Y}(s'(t)Y) = s'(t)D\widehat{exp}|_{s(t)Y}(Y)$$

hence

$$|\eta'(t)| = s'(t)\left|D\widehat{exp}|_{s(t)Y}(Y)\right|\,.$$

But since

$$||\beta'(t)|| = e^{-\sigma(\beta(t))}|\beta'(t)| = e^{-\varphi(exp(tX))}\,|D\widehat{exp}|_{tY}(Y)| = e^{-\varphi}|Y|\,,$$

we have

$$|D\widehat{exp}|_{tY}(Y)| = e^{\varphi(exp(tY))-\varphi}|Y|\,.$$

We have assumed that $s'(t) > 0$ at least for small t, so in order to have $|\eta'(t)| = |Y|$, we need $s'(t) = e^{\varphi - \varphi(exp(tX))}$. This gives the following equations defining s:

$$s'(0) = 1\,, \text{ and } s''(t) = -s'(t) g(grad\ \varphi|_{exp(s(t)X)}, Dexp|_{s(t)X}(s'(t)X))\,,$$

and thus

$$s''(0) = -X(\varphi)\,.$$

Define $\lambda(t) = exp(s(t)X)$. Then

$$s'''(t) = e^{\varphi - \varphi(\lambda(t))} \left(\frac{d\varphi(\lambda(t))}{dt} \right)^2 - e^{\varphi - \varphi(\lambda(t))} \frac{d^2\varphi(\lambda(t))}{dt^2}\,. \tag{2.1.7}$$

Also,

$$\begin{aligned} \frac{d^2\varphi(\lambda(t))}{dt^2} &= \frac{d}{dt} g(grad\ \varphi|_{\lambda(t)}, \lambda'(t)) \\ &= g(\nabla_{\lambda'(t)} grad\ \varphi|_{\lambda(t)}, \lambda'(t)) + g(grad\ \varphi|_{\lambda(t)}, \nabla_{\lambda'(t)}\lambda'(t))\,. \end{aligned}$$

At $t = 0$,

$$\frac{d^2\varphi(\lambda(t))}{dt^2}\Big|_{t=0} = g(\nabla_X grad\ \varphi, X) + g(grad\ \varphi, \nabla_{\lambda'(t)}\lambda'(t)|_{t=0})\,.$$

The curve $\lambda_0(t) = exp(tX)$ is a geodesic in the metric g, thus $\nabla_{\lambda_0'(t)}\lambda_0'(t) = 0$. We conclude that

$$\nabla_{\lambda'(t)}\lambda'(t) = s''(t)\lambda_0'(s(t))\,,$$

and therefore

$$\nabla_{\lambda'(t)}\lambda'(t)|_{t=0} = s''(0)X = -X(\varphi)X\,.$$

Now back to (2.1.7) and setting $t = 0$, we obtain

$$\begin{aligned} s'''(0) &= X(\varphi)^2 - g(\nabla_X grad\ \varphi, X) - g(grad\ \varphi, -X(\varphi)X) \\ &= 2X(\varphi)^2 - Hess(\varphi)(X,X)\,. \end{aligned}$$

If $\tau = s(t)$, then the inverse $t = h(\tau)$ near $\tau = 0$ satisfies

$$h'(0) = 1 \ \ , \ \ h''(0) = X(\varphi) \ \text{ and } \ h'''(0) = Hess(\varphi)(X,X) + X(\varphi)^2\,.$$

We now get

$$\eta(t) = \widehat{exp}(s(t)Y) = \gamma(t) = exp(\mu(t))\,,$$

or

$$exp^{-1} \circ \widehat{exp}(tY) = \mu(h(t))\,.$$

Using equation (2.1.5) we finally obtain

$$\begin{aligned} exp^{-1} \circ \widehat{exp}(tY) &= h(t)Y - \frac{h(t)^2}{2}(grad\ \sigma)^N |Y|^2 + \\ & \frac{h(t)^3}{6}(\{M(-\sigma)Y\}^N - |\{grad\ \sigma\}^N|^2 Y) \\ & +O(t^4)\,, \end{aligned} \tag{2.1.8}$$

with

$$\begin{aligned} h(t) &= t + \frac{1}{2}X(\varphi)^2 t^2 + \frac{1}{6}(Hess(\varphi)(X,X) + X(\varphi)^2)t^3 \\ & +O(t^4)\,. \end{aligned} \tag{2.1.9}$$

We want to write (2.1.8) in powers of t. This involves straightforward calculations and the following identities:

$$grad\ \sigma = e^{-2\varphi}\psi_*(grad\ \varphi)\,, \tag{2.1.10}$$

thus

$$(grad\ \sigma)^N = e^{-2\varphi}\psi_*(grad\ \varphi - \frac{X(\varphi)}{|X|^2}X)\,. \tag{2.1.11}$$

Also

$$M(-\sigma)Y = e^{-2\varphi}\psi_*(M(-\varphi)X) + 2e^{-2\varphi}X(\varphi)\psi_*(grad\ \varphi) - 2\frac{2}{n}e^{-2\varphi}|grad\ \varphi|^2 Y\,,$$

hence

$$(M(-\sigma)Y) = e^{-2\varphi}(\psi_*(M(-\varphi)X)^N + 2e^{-2\varphi}X(\varphi)(\psi_*(grad\ \varphi))^N\,.$$

But

$$\begin{aligned} (\psi_*(M(-\varphi))X)^N &= \psi_*(M(-\varphi)X) - \frac{1}{|Y|^2}g(\psi_*(M(-\varphi)X), Y)Y \\ &= \psi_*(M(-\varphi)X) - \frac{1}{|X|^2}B_g(-\varphi)(X,X)Y\,, \end{aligned} \tag{2.1.12}$$

and

$$M(-\varphi)X = -M(\varphi)X - 2X(\varphi)grad\ \varphi + \frac{2}{n}|grad\ \varphi|^2|X|^2\,. \tag{2.1.13}$$

Therefore

$$(M(-\sigma)Y)^N = -e^{-2\varphi}\psi_*(M(\varphi)X) + e^{-2\varphi}\frac{B_g(\varphi)(X,X)}{|X|^2}Y\,. \tag{2.1.14}$$

Equation (2.1.8) can be finally written as

$$exp^{-1}\circ\widehat{exp}(tY) \;=\; tY + t^2A + t^3B + O(t^4)\,,$$

with

$$\begin{aligned} A &= X(\varphi)Y - \frac{1}{2}|Y|^2 grad\ \sigma\,,\\ 6B &= (M(-\sigma)Y)^N + (Hess(\varphi)(X,X) + X(\varphi)^2)Y - \\ &\quad |(grad\ \sigma)^N|^2|Y|^2Y - \frac{1}{2}X(\varphi)(grad\ \sigma)^N|Y|^2\,. \end{aligned}$$

To derive this last equation, we have used the expression for $h(t)$ as in (2.1.9). The proposition finally follows after replacing the terms $M(-\sigma)Y$, $grad\,\sigma$ and their normal components using equations (2.1.10), (2.1.11) and (2.1.12).

Now let us go back to the infinitesimal deformation of cross-ratio. From (2.1.4) one can directly obtain an expansion in t of the absolute value of the cross-ratio, and we shall establish as the main result in this section the following generalization of it. Let $X_1, X_2, X_3, X_4 \in T_pM$, and let $Z_i(t) = exp^{-1}\circ\widehat{exp}(t\psi_*(X_i))$ for $i = 1,2,3,4$. Our theorem concerns a second order expansion in powers of t of the quantity

$$\begin{aligned} \zeta(t) &= |(Z_1(t), Z_2(t), Z_3(t), Z_4(t))| \\ &= \frac{|Z_1(t)-Z_3(t)||Z_2(t)-Z_4(t)|}{|Z_1(t)-Z_4(t)||Z_2(t)-Z_3(t)|}\,, \end{aligned} \tag{2.1.15}$$

namely

Theorem 2.1.1 *With the notation as before,*

$$\zeta(t) = \zeta(0)(1 + \frac{1}{2}\frac{\zeta''(0)}{\zeta(0)}t^2 + O(t^3))\,,$$

where

$$\begin{aligned}\zeta(0) &= \frac{|X_1 - X_3||X_2 - X_4|}{X_1 - X_4||X_2 - X_3|}\,, \\ \frac{\zeta''(0)}{\zeta(0)} &= \frac{1}{3}\Sigma_4 \frac{B_g(\varphi)(X_i, X_j)(|X_i|^2 + |X_j|^2) - B_g(\varphi)(X_i, X_i)|X_j|^2 - B_g(\varphi)(X_j, X_j)|X_i|^2}{|X_i - X_j|^2} \\ &\quad + \frac{1}{12}(2m(\varphi) + |grad\ \varphi|^2)\Sigma_4 \frac{(|X_i|^2 - |X_j|^2)^2}{|X_i - X_j|^2}\,, \qquad (2.1.16)\end{aligned}$$

and $m(\varphi) = \frac{1}{n}(\Delta\varphi - |grad\ \varphi|^2)$.

Remarks

(1) We have introduced here the following convenient notation:

$$\Sigma_4 A_{ij} = A_{13} + A_{24} - A_{23} - A_{14}\,.$$

(2) Note that the theorem asserts implicitly that $\zeta'(0) = 0$.

Proof: We first note that $Z_i(t)$ can be written as $\psi_*(W_i(t))$, and since ψ_* is a conformal map between tangent spaces, we can as well do all the computations with the W_i's. It is clear that

$$\zeta(0) = \frac{|X_1 - X_3||X_2 - X_4|}{|X_1 - X_4||X_2 - X_3|}\,. \qquad (2.1.17)$$

We compute $\zeta'(0)$ by using logarithmic derivative, which yields

$$2\frac{\zeta'(t)}{\zeta(t)} = \frac{(|W_1 - W_3|^2)'}{|W_1 - W_3|^2} + \frac{(|W_2 - W_4|^2)'}{|W_2 - W_4|^2} - \frac{(|W_1 - W_4|^2)'}{|W_1 - W_4|^2} - \frac{(|W_2 - W_3|^2)'}{|W_2 - W_3|^2}\,. \qquad (2.1.18)$$

At $t = 0$ we get

$$\frac{\zeta'(0)}{\zeta(0)} = \Sigma_4 \frac{g(X_i - X_j, A_i - A_j)}{|X_i - X_j|^2}\,.$$

After a short calculation and using (2.1.6), one obtains

$$g(X_i - X_j, A_i - A_j) = \frac{1}{2}(X_i(\varphi) + X_j(\varphi))|X_i - X_j|^2\,,$$

so that

$$\frac{\zeta'(0)}{\zeta(0)} = \frac{1}{2}\Sigma_4(X_i(\varphi) + X_j(\varphi)) = 0\,,$$

because all the terms cancel out. Hence, $\zeta'(0) = 0$. We differentiate (2.1.18), to obtain

$$2(\frac{\zeta''(t)}{\zeta(t)} - (\frac{\zeta'(t)}{\zeta(t)})^2) = \Sigma_4 \frac{(|W_i - W_j|^2)''}{|W_i - W_j|^2} - \Sigma_4 \frac{\{(|W_i - W_j|^2)'\}^2}{|W_i - W_j|^4} .$$

At $t = 0$ we get

$$\frac{\zeta''(0)}{\zeta(0)} = \Sigma_4 \frac{|A_i - A_j|^2 + 2g(X_i - X_j, B_i - B_j)}{|X_i - X_j|^2} - \frac{1}{2}\Sigma_4 (X_i(\varphi) + X_j(\varphi))^2 .$$

A lengthy calculation again using equations (2.1.6) for the $A_i's$ and $B_i's$ yields the following expression:

$$\begin{aligned} \frac{\zeta''(0)}{\zeta(0)} &= \frac{1}{3}\Sigma_4 \frac{B_g(\varphi)(X_i, X_j)(|X_i|^2 + |X_j|^2) - B_g(\varphi)(X_i, X_i)|X_j|^2 - B_g(\varphi)(X_j, X_j)|X_i|^2}{|X_i - X_j|^2} \\ &\quad + \frac{1}{12}(m(\varphi) + |grad\ \varphi|^2)\Sigma_4 \frac{(|X_i|^2 - |X_j|^2)^2}{|X_i - X_j|^2} , \end{aligned}$$

where

$$m(\varphi) = \frac{1}{n}(\Delta\varphi - |grad\ \varphi|^2) .$$

Thus we finally get

$$\zeta(t) = \zeta(0)(1 + \frac{\zeta''(0)}{2\zeta(0)} t^2 + O(t^3)) , \qquad (2.1.19)$$

where $\zeta(0)$ is as in (2.1.17).

This last equation generalizes the analogous situation in the complex plane. In that case, $\varphi = \log|\psi'|$ and the tensor $B_g(\varphi)$ is given in usual coordinates by (1.2.3). It is interesting to note that in this case, $2m(\varphi) + |grad\ \varphi|^2 = \Delta\varphi = 0$ since φ is harmonic. It is now not difficult to verify that (2.1.16) indeed reduces to the corresponding expansion from (2.1.4).

Finally, and back to the general case, by choosing X_1, X_2, X_3, X_4 appropriately, we can express the Schwarzian tensor in terms of the change in the absolute cross-ratio, as follows: since $B_g(\varphi)$ has trace zero, it is determined by its action on pairs of orthogonal vectors; thus if we let $X_1 = -X_2 = X$ and $X_3 = -X_4 = Y$ with $|X| = |Y| = 1$ and $g(X, Y) = 0$, then the term $\Sigma_4 \frac{(|X_i|^2 - |X_j|^2)^2}{|X_i - X_j|^2} = 0$, and because of the symmetry of the Schwarzian tensor we get that

$$\frac{\zeta''(0)}{\zeta(0)} = \frac{4}{3} B_g(\varphi)(X, Y) .$$

Since for these choices of X_i one has $\zeta(0) = 1$, we conclude

Corollary 2.1.1
$$B_g(\varphi)(X,Y) = \frac{3}{2}\lim_{t\to 0}\frac{\zeta(t)-1}{t^2}\,.$$

In this fashion one can recover the tensor $B_g(\varphi)$ as the first nontrivial term in the infinitesimal deformation of the absolute cross-ratio.

2.2 Cross-ratio and Ricatti equations

A second application of cross-ratio comes up in relation to solutions of a Ricatti equation. Such an equation in the plane is of the form
$$\frac{dy}{dz} = a(z)y^2 + b(z)y + c(z)\,. \tag{2.2.1}$$
It is known in the study of solutions of this last equation, that when the coefficients a, b, c are analytic functions of z, the cross-ratio of any four linearly independent analytic solutions is constant [30]. We will consider the important special case of (2.2.1) when
$$\frac{dy}{dz} - \frac{1}{2}y^2 = 0\,, \tag{2.2.2}$$
whose only solutions are of the form $y = T''/T'$, where T is a Möbius transformation; this is nothing else but the statement that Möbius transformations are the only analytic functions with vanishing Schwarzian derivative.

Our purpose is to establish a similar result about the behavior of the cross-ratio of solutions to a Ricatti equation of the form (2.2.2), now in the context of Riemannian manifolds. We shall use the notion of a Möbius solution as [41] and the cross-ratio introduced in the last section.

The solution T corresponding to y in (2.2.2) is conformal, thus $T^*(g_0) = e^{2\varphi}g_0$. As usual, g_0 is the euclidean metric and $\varphi = \log|T'|$. We can therefore think of y as the gradient of the conformal factor φ (actually, $grad\ \varphi = \overline{y}$).

In order to prove the generalized theorem, we will first establish the result for Möbius transformations in R^n, and then a theorem of Osgood and Stowe will enable us to reduce the case of Möbius transformations in Riemannian manifolds to the euclidean setting.

By Liouville's theorem, Möbius maps are the only conformal transformations in R^n. We shall consider only those which are orientation-preserving. Such mappings are the solutions, say at $t = 1$, of

$$\frac{\partial \psi(x,t)}{\partial t} = V(\psi(x,t)) \ \ , \psi(x,0) = x \, , \tag{2.2.3}$$

where V is a vector field generating a flow of conformal transformations. We will call such vector fields infinitesimal Möbius transformation. Recall from section 1.2 that the only vector fields satisfying this property are of the form

$$V(x) = a + B(x) + 2(c \cdot x)x - |x|^2 c \, , \tag{2.2.4}$$

where $a, b \in R^n$, and B is a linear mapping with

$$SB = \frac{1}{2}(B + B^t) - \frac{1}{n} tr(B) I = 0 \, .$$

Such a mapping B has to be of the form $B = B_1 + \lambda I$ with $B_1^t = -B_1$.

The conformal factor $\varphi = \log |DT|$ can be explicitly computed from (2.2.4). In fact, we have seen in section 1.2 that the single contribution to a nonconstant conformal factor will come from the term $2(c \cdot x)x - |x|^2 c$ in (2.2.4); it corresponds to the composition of inversions $J(J(x) - u(t))$, where $J(x) = \frac{x}{|x|^2}$ and $u(t)$ is any curve with $u(0) = 0$ and $u'(0) = c$. Thus, we find that the gradient of the conformal factor of the mapping $T = \psi_1$ is given by

$$grad \ \varphi(x) = \frac{2(c - |c|^2 x)}{1 - 2c \cdot x + |c|^2 |x|^2} \, . \tag{2.2.5}$$

We introduce the notation $\mu_c(x) = 1 - 2c \cdot x + |c|^2 |x|^2$, and state our first result:

Theorem 2.2.1 *Let T_1, T_2, T_3, T_4 be orientation-preserving Möbius transformations in R^n which have nonconstant conformal factors $\varphi_i = \log |DT_i|$ such that $grad \ \varphi_i \neq grad \ \varphi_j$. Then the cross-ratio $(grad \ \varphi_1, grad \ \varphi_2, grad \ \varphi_3, grad \ \varphi_4)$ is constant.*

Proof: As mentioned before, $grad\ \varphi_i(x) = \mu_i^{-1}(x)(c_i - |c_i|^2 x)$ will be not identically zero iff $c_i \neq 0$, and our hypothesis implies furthermore that $c_i \neq c_j$.

The proof reduces to calculating the absolute cross-ratio and the cosine of the argument of the cross-ratio. We first note that for $grad\ \varphi(x) = 2\mu_c^{-1}(x)(c - |c|^2 x)$

$$|grad\ \varphi(x)|^2 = 4\mu_c^{-2}(x)(|c|^2 + |c|^4|x|^2 - 2|c|c \cdot x) = 4|c|^2\mu_c^{-1}(x)\,.$$

Hence

$$\frac{1}{4}|grad\ \varphi_i - grad\ \varphi_j|^2 = |c_i|^2\mu_{c_i}^{-1} + |c_j|^2\mu_{c_j}^{-1} - 2\mu_{c_i}^{-1}\mu_{c_j}^{-1}(c_i - |c_i|^2 x) \cdot (c_j - |c_j|^2 x)\,,$$

and the right-hand side simplifies after elementary algebraic manipulations to

$$\mu_{c_j}^{-1}\mu_{c_j}^{-1}|c_i - c_j|^2\,.$$

Therefore,

$$\begin{aligned}
|(grad\ \varphi_1, ..., grad\ \varphi_4)| &= \frac{|grad\ \varphi_1 - grad\ \varphi_3||grad\ \varphi_2 - grad\ \varphi_4|}{|grad\ \varphi_1 - grad\ \varphi_4||grad\ \varphi_2 - grad\ \varphi_3|} \\
&= \frac{|c_1 - c_3|\mu_{c_1}^{-1/2}\mu_{c_3}^{-1/2}|c_2 - c_4|\mu_{c_2}^{-1/2}\mu_{c_4}^{-1/2}}{|c_1 - c_4|\mu_{c_1}^{-1/2}\mu_{c_4}^{-1/2}|c_2 - c_3|\mu_{c_2}^{-1/2}\mu_{c_3}^{-1/2}} \\
&= \frac{|c_1 - c_3||c_2 - c_4|}{|c_1 - c_4||c_2 - c_3|}\,.
\end{aligned}$$

This shows that the absolute cross-ratio is constant.

We now analyze the argument of the cross-ratio. We know that

$$\cos(arg(grad\ \varphi_1, ..., grad\ \varphi_4)) = \frac{A \cdot B}{|A||B|}\,,$$

where A and B are given by

$$\begin{aligned}
A &= \frac{X_1 - X_4}{|X_1 - X_4|^2} - \frac{X_3 - X_4}{|X_3 - X_4|^2}\,, \\
B &= \frac{X_2 - X_4}{|X_2 - X_4|^2} - \frac{X_3 - X_4}{|X_3 - X_4|^2}\,.
\end{aligned}$$

For the sake of brevity, we have used the notation $X_i = grad\ \varphi_i$.

We compute first

$$|A|^2 = |X_1 - X_4|^{-2} + |X_3 - X_4|^{-2} - 2|X_1 - X_4|^{-2}|X_3 - X_4|^{-2}(X_1 - X_4) \cdot (X_3 - X_4)\,,$$

thus

$$\begin{aligned}|X_1-X_4|^2|X_3-X_4|^2|A|^2 &= |X_1-X_4|^2+|X_3-X_4|^2-2(X_1-X_4)\cdot(X_3-X_4)\\ &= 4\frac{|c_1-c_4|^2}{\mu_{c_1}\mu_{c_4}}+4\frac{|c_3-c_4|^2}{\mu_{c_3}\mu_{c_4}}-2(X_1-X_4)\cdot(X_3-X_4)\,.\end{aligned}$$

But

$$(X_1-X_4)\cdot(X_3-X_4)=4(\frac{c_1-|c_1|^2x}{\mu_{c_1}}-\frac{c_4-|c_4|^2x}{\mu_{c_4}})\cdot(\frac{c_3-|c_3|^2x}{\mu_{c_3}}-\frac{c_4-|c_4|^2x}{\mu_{c_4}})\,,$$

which after some simplifications can be written as

$$2\mu_{c_1}^{-1}\mu_{c_3}^{-1}\mu_{c_4}^{-1}(|c_1-c_4|^2\mu_{c_3}+|c_3-c_4|^2\mu_{c_1}-|c_1-c_3|^2\mu_{c_4})\,.$$

Hence

$$|X_1-X_4|^2|X_3-X_4|^2|A|^2=4\mu_{c_1}^{-1}\mu_{c_3}^{-1}|c_1-c_3|^2\,,$$

and so

$$|A|^2=\frac{1}{4}\frac{|c_1-c_3|^2}{\mu_{c_1}\mu_{c_3}}\frac{\mu_{c_1}\mu_{c_4}}{|c_1-c_4|^2}\frac{\mu_{c_3}\mu_{c_4}}{|c_3-c_4|^2}=\frac{1}{4}\mu_{c_4}\frac{|c_1-c_3|^2}{|c_1-c_4|^2|c_3-c_4|^2}\,.$$

Similarly,

$$|B|^2=\frac{1}{4}\mu_{c_4}\frac{|c_2-c_3|^2}{|c_2-c_4|^2|c_3-c_4|^2}\,.$$

We now calculate

$$A\cdot B=(\frac{X_1-X_4}{|X_1-X_4|^2}-\frac{X_3-X_4}{|X_3-X_4|^2})\cdot(\frac{X_2-X_4}{|X_2-X_4|^2}-\frac{X_3-X_4}{|X_3-X_4|^2})\,.$$

One of the summands in the expansion of the right-hand side is

$$\begin{aligned}\frac{X_1-X_4}{|X_1-X_4|^2}\cdot\frac{X_2-X_4}{|X_2-X_4|^2} &= |X_1-X_4|^{-2}|X_2-X_4|^{-2}(X_1-X_4)\cdot(X_2-X_4)\\ &= 2|X_1-X_4|^{-2}|X_2-X_4|^{-2}\mu_{c_1}^{-1}\mu_{c_2}^{-1}\mu_{c_4}^{-1}(|c_1-c_4|^2\mu_{c_2}+\\ &\quad |c_2-c_4|^2\mu_{c_1}-|c_1-c_2|^2\mu_{c_4})\,.\end{aligned}$$

Hence

$$\frac{X_1-X_4}{|X_1-X_4|^2}\cdot\frac{X_2-X_4}{|X_2-X_4|^2}=\frac{1}{8}\left(\frac{\mu_{c_1}\mu_{c_4}}{|c_1-c_4|^2}+\frac{\mu_{c_2}\mu_{c_4}}{|c_2-c_4|^2}-\frac{|c_1-c_2|^2\mu_{c_4}^2}{|c_1-c_4|^2|c_2-c_4|^2}\right)\,,$$

and by symmetry, two of the other summands in $A \cdot B$ are

$$-\frac{1}{8}(\frac{\mu_{c_2}\mu_{c_4}}{|c_2-c_4|^2}+\frac{\mu_{c_3}\mu_{c_4}}{|c_3-c_4|^2}-\frac{|c_2-c_3|^2\mu_{c_4}^2}{|c_2-c_4|^2|c_3-c_4|^2})$$

and

$$-\frac{1}{8}(\frac{\mu_{c_1}\mu_{c_4}}{|c_1-c_4|^2}+\frac{\mu_{c_3}\mu_{c_4}}{|c_3-c_4|^2}-\frac{|c_1-c_3|^2\mu_{c_4}^2}{|c_1-c_4|^2|c_3-c_4|^2})\,.$$

The last summand in $A \cdot B$ is

$$|X_3-X_4|^{-2}=\frac{1}{4}\mu_{c_3}\mu_{c_4}|c_3-c_4|^{-2}\,.$$

Thus, putting these four terms together and after some cancellations, we conclude

$$A\cdot B=\frac{1}{8}\mu_{c_4}^2(\frac{|c_1-c_3|^2}{|c_1-c_4|^2|c_3-c_4|^2}+\frac{|c_2-c_3|^2}{|c_2-c_4|^2|c_3-c_4|^2}-\frac{|c_1-c_2|^2}{|c_1-c_4|^2|c_2-c_4|^2})\,.$$

Therefore, we are finally able to obtain

$$\begin{aligned}\cos(arg(X_1,X_2,X_3,X_4)) &= \frac{A\cdot B}{|A||B|}\\ &= \frac{|c_1-c_3||c_2-c_3|}{|c_1-c_4||c_2-c_4|}(|c_1-c_4|^2|c_2-c_3|^2+\\ &\quad |c_1-c_3|^2|c_2-c_4|^2-|c_1-c_2|^2|c_3-c_4|^2)\,.\end{aligned}$$

This proves our theorem.

From this and the work in [41] on Möbius transformations on Riemannian manifolds, it is now easy to establish the result in that context. Let (M,g) be an n-dimensional Riemannian manifold with metric g. We consider four orientation-preserving Möbius transformations T_1,T_2,T_3,T_4 on M, which are generated as before by vector fields V_1,V_2,V_3,V_4. The functions $u_i=|DT_i|^{-1}$ are solutions of the equation $Hess(u)=\frac{\Delta u}{n}g$. The family of all solutions to this last equation induces a warped product decomposition of M as $Q\times_f P$ such that the submanifold Q, which is integral to the gradients of all the elements in the family, is of constant curvature. Furthermore, the infinitesimal Möbius transformations $V_1,...,V_4$ are tangent to the leaves $Q\times\{p\}$, which in turn are mapped into themselves by $T_1,...,T_4$.

On the other hand, such a space Q is locally isometric to an open subset of R^m with a metric g_1 which is conformal to the euclidean metric g_0. If we write $g_1=e^{2\sigma}g_0$, then $B_{g_0}(\sigma) = 0$ as well, and hence the Möbius groups in the metrics g_1 and g_0 coincide. The images in R^m of the vector fields $V_1, ..., V_4$ under the local isometry will generate Möbius transformations $R_1, ..., R_4$, whose conformal factors will have by Theorem 2.2.1 constant cross-ratio in the metric g_0. Since the cross-ratio is clearly a conformal invariant (a conformal change in the metric will just scale, at each point, the 2-sphere through the four vector fields defining the cross-ratio), we conclude that the cross-ratio in Q of $grad(\log u_1), ..., grad(\log u_4)$ is constant. But these gradients are tangent to Q, and therefore the 2-sphere through them lies in the tangent space to Q. Hence their cross-ratio in Q is the same as their cross-ratio in M. We have thus shown

Theorem 2.2.2 *Let (M, g) be a Riemannian manifold, and let T_1, T_2, T_3, T_4 be orientation-preserving Möbius transformations of M with nonconstant conformal factors $\varphi_i = \log |DT_i|$. Then the cross-ratio $(grad\ \varphi_1, grad\ \varphi_2, grad\ \varphi_3, grad\ \varphi_4)$ is constant.*

This theorem can be regarded as a generalization of the original result about solutions to a Ricatti equation. Indeed, the conformal factors $\varphi_i = \log |DT_i|$ are Schwarzian, i.e., $B_g(\varphi_i) = 0$, which can be thought of as a Ricatti equation in $grad\ \varphi_i$.

2.3 Holomorphicity of the cross-ratio

This section will be devoted to a brief study of the cross-ratio as a complex- valued map on M which is determined by four vector fields on M. We believe that the following two questions deserve attention:

(1) on a complex manifold M, what kind of complex-valued functions on M can be expressed as cross-ratios; in particular, which holomorphic ones can be so represented;

(2) on complex manifolds, characterize 4-tuples of (holomorphic or meromorphic) vector fields which give rise to holomorphic or meromorphic functions. The second question has eluded us almost completely, even in C^n. We have found a conceptually rather simple (but not easy to present formally) sufficient condition for a 4-tuple of holomorphic vector fields to have a holomorphic cross-ratio. But we have not been able to show its necessity, so instead of discussing this rather unsatisfactory result,

we will center our attention on a partial answer to the first problem, namely when M is a Riemann surface.

Theorem 2.3.1 *Every meromorphic function on a Riemann surface is a cross-ratio.*

Proof: On M, a meromorphic vector field V has in terms of a local uniformizer z an invariant expression of the form $A(z)\partial_z$. Let f be a meromorphic function on M. We want to find (holomorphic) vector fields V_1, V_2, V_3, V_4 such that in terms of the local uniformizer,

$$f(z) = \left(\frac{A_1(z) - A_3(z)}{A_1(z) - A_4(z)}\right)\left(\frac{A_2(z) - A_4(z)}{A_2(z) - A_3(z)}\right);$$

here $A_i(z)\partial_z$ is the local representation of V_i.

This last calls for an explanation: we have defined cross-ratio on the real tangent space of a manifold, whereas suddenly we are talking about the complexified tangent space on M. This causes no problem, since it is not difficult to convince oneself that the canonical identification of the real tangent space with the holomorphic part of the complexification takes the cross-ratio to the expression in our last equation.

Because of the relation between vector fields and differentials, and since $(V_1 - V_3) + (V_2 - V_4) = (V_1 - V_4) + (V_2 - V_3)$, the problem is equivalent to finding meromorphic differentials $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ such that

$$f = \frac{(1/\zeta_1)(1/\zeta_2)}{(1/\zeta_3)(1/\zeta_4)} = \frac{\zeta_3\zeta_4}{\zeta_1\zeta_2},$$

with the condition that

$$\frac{1}{\zeta_1} + \frac{1}{\zeta_2} = \frac{1}{\zeta_3} + \frac{1}{\zeta_4}.$$

In other words, we want to solve

$$f = \frac{\zeta_3\zeta_4}{\zeta_1\zeta_2} = \frac{\zeta_3 + \zeta_4}{\zeta_1 + \zeta_2}. \tag{2.3.1}$$

We will show that solutions to (2.3.1) can be obtained from solutions to

$$\frac{1}{f} = 1 - gh, \tag{2.3.2}$$

where g, h are meromorphic functions on M not identically zero, and vice-versa. Since (2.3.2) always has solutions (just take any g not identically 0 or 1, and then choose $h = g^{-1}(1 - f^{-1})$), we conclude that a cross-ratio representation is always possible.

For the proof, assume first that g, h solve (2.3.2). Let $g_1 = 1 - g$ and $h_1 = 1 - h$, and choose ζ_3 to be any nonzero meromorphic differential. Define

$$\zeta_1 = g_1 \zeta_3 \ \ , \ \ \zeta_2 = h_1 \zeta_3$$

and

$$\zeta_4 = \frac{\zeta_1\zeta_2\zeta_3}{\zeta_1\zeta_3 + \zeta_2\zeta_3 - \zeta_1\zeta_2} = \frac{g_1 h_1}{g_1 + h_1 - g_1 h_1}\zeta_3 \, .$$

The expression of ζ_4 in terms of $\zeta_1, \zeta_2, \zeta_3$ is equivalent to the second equality in (2.3.1), and we now check

$$\frac{\zeta_3\zeta_4}{\zeta_1\zeta_2} = \frac{1}{g_1 + h_1 - g_1 h_1} = \frac{1}{1 - gh} = f \, .$$

On the other hand, assume that we have a solution of (2.3.1). Then, as mentioned before, the second equality gives

$$\zeta_4 = \frac{\zeta_1\zeta_2\zeta_3}{\zeta_1\zeta_3 + \zeta_2\zeta_3 - \zeta_1\zeta_2} \, ,$$

and therefore

$$f = \frac{\zeta_3^2}{\zeta_1\zeta_3 + \zeta_2\zeta_3 - \zeta_1\zeta_2} = \frac{1}{(\zeta_1/\zeta_3) + (\zeta_2/\zeta_3) - (\zeta_1/\zeta_3)(\zeta_2/\zeta_3)} \, .$$

This gives a solution of (2.3.2) with $g = 1 - (\zeta_1/\zeta_3)$ and $h = 1 - (\zeta_2/\zeta_3)$.

Chapter 3

Univalence criteria

3.1 The theorem of Osgood and Stowe

As mentioned in the introduction, the main purpose of this chapter is to derive several classical and a few new univalence criteria from the general theorem of Osgood and Stowe. Before stating their result, we need a few more definitions and preliminary results.

By $||B_g(\varphi)||$ we mean the norm of the Schwarzian tensor $B_g(\varphi)$ with respect to g, as a bilinear form on each tangent space, that is,

$$||B_g(\varphi)|| = max\{|B_g(\varphi)(X,Y)| \ : \ |X| = |Y| = 1 \ \} .$$

In cases, we will need to consider the norm of $||B_g(\varphi)||$ in a metric $\hat{g} = e^{2\sigma}g$ conformal to g. Then

$$||B_g(\varphi)||_{\hat{g}} = e^{-2\sigma}||B_g(\varphi)|| .$$

Recall that a metric $\hat{g} = e^{2\varphi}g$ is said to be Möbius (with respect to g) if $B_g(\varphi) = 0$. The most general Möbius metric on a subset of R^n conformal to the euclidean metric has

$$\varphi(x) = -\log(a|x|^2 + b \cdot x + c) \ , \ a, c \in R \ , b \in R^n .$$

These metrics have constant curvature $4ac - |b|^2$. In particular, the Poincaré metric

$$\frac{1}{1-|z|^2}|dz|$$

on the disc and the spherical metric

$$\frac{2}{1+|x|^2}|dx|$$

on $R^n \cup \{\infty\}$ are Möbius.

Recall also that on a general manifold (M, g), the substitution $u = e^{-\varphi}$ converts the equations $B_g(\varphi) = 0,\ B_g(\varphi) = p$ into the linear equations

$$\begin{aligned} Hess(u) &= \frac{\Delta u}{n} g \\ Hess(u) + up &= \frac{\Delta u}{n} g \end{aligned} \tag{3.1.1}$$

respectively. Osgood and Stowe define $U(M)$ to be the space of all solutions to the first of these last two equations. If $u \in U(M)$ with $u > 0$, then $B_g(-\log u) = 0$.

Using stereographic coordinates we can write the round metric g_1 on S^n as $4(1 + |x|^2)^{-2}|dx|^2 = e^{2\varphi_0} g_0$ on $R^n \cup \{\infty\}$. Since $B_{g_0}(\varphi_0) = 0$, we find from the addition formula (1.2.4) and the solutions in R^n that the general solution to $B_{g_1}(\varphi) = 0$ on S^n is of the form

$$\varphi(x) = -\log \frac{A|x|^2 + B \cdot x + C}{|x|^2 + 1}\ ,\ A, C \in R,\ B \in R^n\ ,$$

in these coordinates, and a general $u \in U(S^n)$ is of the form

$$u(x) = \frac{A|x|^2 + B \cdot x + C}{|x|^2 + 1}\ .$$

Then $u^{-2} g_1$ has curvature $AC - \frac{1}{4}|B|^2$. We see from this that if $u^{-2} g_1$ is flat, then u vanishes at precisely one point in S^n and hence is otherwise of one sign. This important fact will be used later and Osgood and Stowe state it as the following

Lemma 3.1.1 *For each $p \in S^n$ there is a $u \in U(S^n)$ such that $u(p) = 0,\ u > 0$ on $S^n/\{p\}$ and $u^{-2} g_1$ is flat.*

We need one more formula. For a metric g on M let $k = (n(n-1))^{-1} scal(g)$, where *scal* is the scalar curvature. If $\hat{g} = e^{2\varphi} g$ and $\hat{k}$ is the corresponding quantity, then

$$\hat{k} = e^{-2\varphi}(k - \frac{2}{n}\Delta\varphi - (\frac{n-2}{n})|grad\ \varphi|^2)\ . \tag{3.1.2}$$

With this, we now present the result in [42].

Theorem 3.1.1 *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and $\psi : (M, g) \to (S^n, g_1)$ a conformal local diffeomorphism. Suppose that the scalar curvature of M is bounded above by $n(n-1)K$ for some $K \in R$, and that any two points in M can be joined by a geodesic of length $< \delta$ for some $0 < \delta \leq \infty$. If*

$$||S_g(\psi)|| \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}K$$

then ψ is injective.

Proof: Let $\varphi = \log |D\psi|$, so that $\psi^*(g_1) = e^{2\varphi} g = \hat{g}$. Let $x \in M, p = \psi(x)$ and choose a function $u \in U(S^n)$ vanishing at p, which is otherwise positive and is such that $u^{-2}g_1$ is flat. Define

$$w = (u \circ \psi)e^{-\varphi}$$

on M. Then

$$w^{-2}g = \psi^*(u^{-2}g_1)$$

is a flat metric on $M/\psi^{-1}(p)$. Using the addition formula (1.2.4), we find that

$$\begin{aligned} B_g(-\log w) &= B_g(\varphi - \log(u \circ \psi)) \\ &= B_g(\varphi) + B_{\hat{g}}(-\log(u \circ \psi)) \\ &= S_g(\psi) + \psi^*(B_{g_1}(-\log u)) = S_g(\psi)\,, \end{aligned}$$

because $B_{g_1}(-\log u) = 0$. Hence from (3.1.1) we may write

$$Hess(w) = -wS_g(\psi) + \frac{\Delta w}{n} g\,. \tag{3.1.3}$$

(This last equation holds on all of M.)

Let k be $(n(n-1))^{-1}$ times the scalar curvature of g. Since the metric $w^{-2}g$ is flat, equation (3.1.3) gives

$$\begin{aligned} 0 &= k - \frac{2}{n}\Delta(-\log w) - \frac{n-2}{n}|grad \log w|^2 \\ &= k + \frac{2}{n}\frac{\Delta w}{w} - \frac{|grad\ w|^2}{w^2}\,. \end{aligned} \tag{3.1.4}$$

The assumption on the scalar curvature then implies

$$\frac{\Delta w}{n} \geq \frac{w}{2}(K + \frac{|grad\ w|^2}{w^2})\,. \tag{3.1.5}$$

Now let $\gamma : [0, l) \to M, l \leq \delta$, be a unit speed geodesic for g with $\gamma(0) = x$. Write $w(t)$ for w evaluated along γ. Then $w(0) = 0$ and $w(t) > 0$ for small positive t. From (3.1.3), (3.1.5) and the bound on $||S_g(\psi)||$ we obtain, whenever $w(t) > 0$,

$$\begin{aligned} w'' &= Hess(w)(\dot\gamma, \dot\gamma) = -wS_g(\psi)(\dot\gamma, \dot\gamma) + \frac{\Delta w}{n} \\ &\geq -w\left(\frac{2\pi^2}{\delta^2} - \frac{1}{2}K\right) + \frac{1}{2}w\left(-K + (\frac{w'}{w})^2\right) \\ &= -\frac{2\pi^2}{\delta^2}w + \frac{1}{2}\frac{(w')^2}{w}\,. \end{aligned}$$

We write this as

$$(w^{1/2})'' \geq -\frac{\pi^2}{\delta^2}w^{1/2}\,. \tag{3.1.6}$$

To summarize, $w(0) = 0,\ w(t) > 0$ for sufficiently small positive t and (3.1.6) holds whenever $w(t) > 0$. Since $l \leq \delta$, the simplest Sturm comparison theorem (see e.g. [15], p.33) guarantees that $w(t)$ cannot vanish again. But then $\psi(\gamma(t)), t \in (0, l)$ cannot equal $\psi(x)$ and the theorem is proved.

We point out that the same theorem could be stated replacing (S^n, g_1) by (R^n, g_0) or H^n with its metric of constant negative curvature. This follows from the transformation law (1.2.4) and the fact that both g_1 (in stereographic coordinates) and the hyperbolic metric are Möbius with respect to the euclidean metric. Finally, let $k(x)$ be the scalar curvature of g at $x \in M$. It is easy to see that the proof given by Osgood and Stowe works equally well only assuming that at each point in M the norm of the Schwarzian derivative of ψ is bounded above by

$$\frac{2\pi^2}{\delta^2} - \frac{k(x)}{2n(n-1)}.$$

This is the form of the theorem which we shall use in order to establish the various injectivity criteria in the unit disc.

3.2 Injectivity criteria in the unit disc

Throughout this section, D will denote the open unit disc in the plane. We will apply Theorem 3.1.2 to D with metrics conformal to the euclidean metric g_0 which are of the form

$$g = \frac{e^{2\sigma}}{(1-|z|^2)^{2t}}\, g_0 \quad , t > 0\,. \tag{3.2.1}$$

Epstein's general theorem of univalence in [22] will follow by setting $t = 1$, and an important case will be when σ is harmonic. In order to apply Theorem 3.1.2, we will require g to have nonpositive curvature and also impose a growth condition on the coefficient $e^{\sigma}(1-|z|^2)^{-t}$ that will ensure that any two points in D can be joined by a geodesic in the metric g. We start out by establishing the following

Lemma 3.2.1 *Let g be as in (3.2.1). If for some $0 < r < 1$*

$$|\sigma_z(z)| \le \frac{t|z|}{1-|z|^2} \tag{3.2.2}$$

for all $r \le |z| < 1$, then any two points in D can be joined by a geodesic in the metric g.

We point out that when $t = 1$, (3.2.2) is esentially one of the conditions Epstein imposes to obtain his general result. It is interesting to note that in his work, this condition is required to assure completeness of certain surfaces in hyperbolic space. Here, it almost ensures the completeness of the metric g (the boundary of D might still be at finite distance). The other condition that Epstein imposes is the negativity of the curvature of g (the sign of this curvature is reflected on certain bounds on the principal curvatures of the surfaces mentioned above). In our case, it is needed to make sure that an inequality on the Schwarzian derivative as in Theorem 3.1.2 is possible (remember that δ might be infinite).

Proof: Let $\eta = \sigma - t\log(1-|z|^2)$. Then (3.2.2) guarantees that the radial derivative of η is nonegative for $|z| \ge r$. Given now two points $x, y \in D$, we seek a geodesic in the metric g joining x and y. Let $d = inf L_g(\gamma)$, where the infimum is taken over all smooth curves γ in D that join x to y, and $L_g(\gamma)$ is the length of γ in g.

Under the hypothesis of the lemma, the conformal factor e^η eventually increases as one approaches ∂D. Hence, given $\epsilon > 0$, there exists a compact set $K \subset D$ such that any curve γ joining x to y with $L_g(\gamma) < d + \epsilon$ is completely contained in K. We can therefore find a minimizing sequence $\{\gamma_n\}$ converging to the desired geodesic.

We now state the main result in this section.

Theorem 3.2.1 *Let g as in (3.2.1) have negative curvature and satisfy (3.2.2). Let $\delta \le \infty$ be the diameter of (M, g), and let ψ be analytic and locally injective in D. If*

$$\left|\frac{(1-|z|^2)^2(\sigma_{zz} - \sigma_z^2 - \frac{1}{2}\{\psi, z\}) - 2t\bar{z}(1-|z|^2)\sigma_z + t(1-t)\bar{z}^2}{t + (1-|z|^2)^2\sigma_{z\bar{z}}}\right| \le$$

$$1 + \frac{\pi^2 e^{2\sigma}(1-|z|^2)^{2(1-t)}}{\delta^2(t + (1-|z|^2)\sigma_{z\bar{z}})} \tag{3.2.3}$$

then ψ is univalent.

We remark that in general $\delta = \infty$ for $t \ge 1$. On the other hand, for $t < 1$ and certain choices of σ, the diameter δ will be finite. We will come back to this point later.

Proof: As mentioned before, we shall derive (3.2.3) from the theorem of Osgood and Stowe by computing the Schwarzian derivative of ψ in the metric g. The scalar curvature of g is given by

$$k = -8e^{-2\eta}\eta_{z\bar{z}} = -8e^{2\eta}(t + (1-|z|^2)^2\sigma_{z\bar{z}}),$$

where as before, $\eta = \sigma - t\log(1-|z|^2)$. We also remind the reader that in this case, the scalar curvature equals twice the standard Gaussian curvature.

Let $\varphi = \log|\psi'|$. Then,

$$\psi^*(g_0) = e^{2\varphi}g_0 = e^{2(\varphi-\eta)}g\,,$$

and we therefore have to compute $B_g(\varphi - \eta)$. Using the addition formula (1.2.4), we have

$$B_g(\varphi - \eta) = B_g(-\eta) + B_{g_0}(\varphi)\,,$$

and since

$$0 = B_g(\eta - \eta) = B_g(-\eta) + B_{g_0}(\eta)\,,$$

we conclude that

$$B_g(\varphi - \eta) = B_{g_0}(\varphi) - B_{g_0}(\eta)\,.$$

Now, by definition

$$||B_g(\varphi - \eta)|| = e^{-2\eta}||B_g(\varphi - \eta)||_{g_0} = e^{-2\eta}||B_{g_0}(\varphi) - B_{g_0}(\eta)||_{g_0}\,. \qquad (3.2.4)$$

Computing in standard coordinates, $B_{g_0}(\varphi) - B_{g_0}(\eta)$ is a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix},$$

and its euclidean norm is $|\alpha + i\beta|$. By (1.2.3), $B_{g_0}(\varphi)$ will be represented by $\overline{\{\psi, z\}}$ and $B_{g_0}(\eta)$ will be given by $A + iB$, where

$$\begin{aligned} A &= \eta_{xx} - \eta_x^2 - \frac{1}{2}(\eta_{xx} + \eta_{yy} - \eta_x^2 - \eta_y^2)\,, \\ B &= \eta_{xy} - \eta_x\eta_y\,. \end{aligned}$$

A straightforward calculation yields

$$\overline{A + iB} = 2\sigma_{zz} - 2\sigma_z^2 - \frac{4t\bar{z}\sigma_z}{1 - |z|^2} + \frac{2t(1-t)\bar{z}^2}{(1 - |z|^2)^2}\,.$$

Theorem 3.1.2 as in the remark after its statement in the last section reads as

$$||S_g(\psi)|| \le \frac{2\pi^2}{\delta^2} + 2\frac{t + (1 - |z|^2)\sigma_{z\bar{z}}}{e^{2\eta}}\,.$$

From this, equation (3.2.4) and the last computations, we obtain (3.2.3).

If in (3.2.3) we let $t = 1$, then setting $\delta = \infty$ we obtain

Corollary 3.2.1 *(Epstein) Let g as in (3.2.1) have negative curvature and satisfy (3.2.2). If ψ is analytic and locally injective in D and satisfies*

$$\left|\frac{(1 - |z|^2)^2(\sigma_{zz} - \sigma_z^2 - \frac{1}{2}\{\psi, z\}) - 2\bar{z}(1 - |z|^2)\sigma_z}{1 + (1 - |z|^2)^2\sigma_{z\bar{z}}}\right| \le 1 \qquad (3.2.5)$$

then ψ is univalent.

As Epstein points out, an important case of (3.2.5) is when σ is harmonic, i.e., $\sigma = \log|h'|$ for some h analytic in D. The inequality (3.2.5) then becomes

$$\left|\{\psi, z\} - \{h, z\} + \frac{2\bar{z}}{(1-|z|^2)}\frac{h''(z)}{h'(z)}\right| \leq \frac{2}{(1-|z|^2)^2}\,.$$

We would like to let σ be harmonic in (3.2.3). Then for $\eta = \sigma - t\log(1-|z|^2)$ the metric $g = e^{2\eta}g_0$ has in any case negative curvature, but we need σ to satisfy (3.2.2) for the existence of a $\delta \leq \infty$. With $\sigma = \log|h'|$ as before, (3.2.2) translates to

$$|\frac{h''(z)}{h'(z)}| \leq \frac{2t|z|}{1-|z|^2}\,. \tag{3.2.6}$$

Assuming this, we conclude

Corollary 3.2.2 *Let ψ be analytic and locally injective in D. If*

$$\left|\{\psi, z\} - \{h, z\} + \frac{2t\bar{z}}{(1-|z|^2)}\frac{h''(z)}{h'(z)} - \frac{2t(1-t)\bar{z}^2}{(1-|z|^2)^2}\right| \leq \frac{2t}{(1-|z|^2)^2} + \frac{2\pi^2 e^{2\eta}}{\delta^2} \tag{3.2.7}$$

then ψ is univalent.

In (3.2.7), we now let $h(z) = z$. As mentioned before, $\delta = \infty$ if $t \geq 1$, but for $t < 1$

$$\delta = 2\int_0^1 \frac{dx}{(1-x^2)^t}\,.$$

This integral can be expressed in terms of the Γ-function as

$$\delta = \pi^{\frac{1}{2}}\frac{\Gamma(1-t)}{\Gamma(\frac{3}{2}-t)}\,.$$

Hence we conclude

Corollary 3.2.3 *Let ψ be analytic and locally univalent in D. If either*

$$\left|\{\psi, z\} - \frac{2t(1-t)}{(1-|z|^2)^2}\bar{z}^2\right| \leq \frac{2t}{(1-|z|^2)^2} \quad, t \geq 1 \tag{3.2.8}$$

or

$$\left|\{\psi, z\} - \frac{2t(1-t)}{(1-|z|^2)^2}\bar{z}^2\right| \leq \frac{2t}{(1-|z|^2)^2} + \frac{2\pi}{(1-|z|^2)^{2t}}\left(\frac{\Gamma(\frac{3}{2}-t)}{\Gamma(1-t)}\right)^2 \quad, t < 1 \tag{3.2.9}$$

then ψ is injective.

We make two remarks on this corollary. If we let $t = 2$ in (3.2.8), then we obtain

$$\left|\{\psi, z\} + \frac{4\bar{z}^2}{(1-|z|^2)^2}\right| \le \frac{4}{(1-|z|^2)^2} \tag{3.2.10}$$

as a sufficient condition for ψ to be univalent. This can be used to give another proof of the criterion announced by Pokornyi [44] and proved by Nehari [38], namely that

$$|\{\psi, z\}| \le \frac{4}{1-|z|^2} \tag{3.2.11}$$

implies the univalence of ψ. In fact, if (3.2.11) holds, then

$$\left|\{\psi, z\} + \frac{4\bar{z}^2}{(1-|z|^2)^2}\right| \le \frac{4}{1-|z|^2} + \frac{4|z|^2}{(1-|z|^2)^2} = \frac{4}{(1-|z|^2)^2}\,,$$

i.e., (3.2.10) holds.

The inequality (3.2.9) interpolates the criteria of Nehari:

$$|\{\psi, z\}| \le \frac{\pi^2}{2} \quad \text{and} \quad |\{\psi, z\}| \le \frac{2}{(1-|z|^2)^2}\,,$$

which are obtained from (3.2.9) as limiting cases when $t \to 0$ and $t \to 1$.

Both of Nehari's criteria are sharp, with extremal functions that are geometrically simple. I therefore consider the following problem of interest: determine whether or not the interpolating criteria are sharp as well, and if they are, try to choose simple extremal functions varying smoothly in t.

Let now $\psi = h$ in (3.2.7), and assume that the metric ψ satisfies (3.2.6). We then conclude

Corollary 3.2.4 *If ψ satisfies*

$$\left|z\frac{\psi''}{\psi'}(1-|z|^2) - (1-t)|z|^2\right| \le 1\,, \tag{3.2.12}$$

then ψ is injective.

Inequalities (3.2.10) and (3.2.12) resemble two criteria of Ahlfors [1] that imply the injectivity of ψ, namely

$$\left|\{\psi, z\} - \frac{2c(1-c)\bar{z}^2}{(1-|z|^2)^2}\right| \le \frac{2t|c|}{(1-|z|^2)^2} \tag{3.2.13}$$

with $|c-1| \leq t < 1, c \in C$, and

$$\left| z\frac{\psi''}{\psi'}(1-|z|^2) + c|z|^2 \right| \leq t < 1 \tag{3.2.14}$$

with $|c| \leq t, c \in C$.

Ahlfors actually shows that either condition implies that ψ has a $\frac{1+t}{1-t}$-quasiconformal extension to C. These two criteria are obtained as corollaries in [9]. The difficulty in deriving (3.2.13) and (3.2.14) from our work lies in the fact that c is complex. On the other hand, (3.2.13) and (3.2.14) can be used to give the following application of (3.2.10) and (3.2.12).

Theorem 3.2.2 *Let ψ be analyic and locally injective in D.*

(A) If ψ satisfies (3.2.12) with $|t-1| \leq 1$, then

$$\phi(z) = \int_0^z (\psi'(\zeta))^\alpha d\zeta$$

is univalent for all $|\alpha| < 1$.

(B) If ψ satisfies (3.2.10), then any solution of $\{\phi, z\} = \alpha\{\psi, z\}$ is univalent for all $|\alpha| < 1$.

Proof: Under the assumption in (A), the function ϕ satisfies (3.2.14) if $|\alpha| < 1$. Similarly, if the hypothesis in (B) holds, then ϕ will satisfy (3.2.13) for $|\alpha| < 1$.

Unfortunately, we were not able to use this to answer the question posed by Pfaltzgraff in his article on the univalence of the integral of $(\psi')^\alpha$ [43]. In that paper, the author shows that a univalent function ψ in D gives rise to univalent functions ψ_α defined by

$$\psi_\alpha(z) = \int_0^z (\psi'(\zeta))^\alpha d\zeta$$

for all $|\alpha| \leq \frac{1}{4}$. Royster exhibited counterexamples for any $|\alpha| > \frac{1}{3}, \alpha \neq 1$, and thus the question of the univalence of ψ_α remains open for $\frac{1}{4} < |\alpha| \leq \frac{1}{3}$.

As a last application in this section, we let $t = 0$ in (3.2.7). In order to obtain a valid criterion we have to change the condition that ensures the existence of a $\delta \leq \infty$. Such a geodesic diameter will exist if for instance, $h(D)$ is convex; for then h is univalent and an isometry between (D, g) and $(h(D), g_0)$. Thus, δ equals the euclidean diameter of $h(D)$. We can take h to be a Möbius transformation $h(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$ and $|d| > |c|$. Then we obtain as a sufficient condition for univalence the inequality

$$|\{\psi, z\}| \leq \frac{\pi^2}{2} \frac{(|d|^2 - |c|^2)^2}{|cz+d|^4} . \tag{3.2.15}$$

3.3 The simply-connected case

Here, we shall derive from the theorem of Osgood and Stowe a sufficient condition for the univalence of a locally schlicht analytic map defined on a simply-connected domain D_1. This condition will come as a counterpart to the necessary condition for such global univalence established in [14], namely

$$|\frac{1}{\pi} U_\psi(z,z) + l(z,z)| \leq K(z,\bar{z}) . \tag{3.3.1}$$

The terms involved are defined as follows: let

$$U_\psi(z,\zeta) = \frac{\partial^2}{\partial\zeta\partial z} \log \frac{\psi(z) - \psi(\zeta)}{z - \zeta} ,$$

so that

$$U_\psi(z,z) = -\frac{1}{6}\{\psi, z\} .$$

Let $h(z,\zeta)$ be the Green's function for the Dirichlet problem in D_1. Then

$$K(z,\bar{\zeta}) = -\frac{2}{\pi} \frac{\partial^2}{\partial z \partial\bar{\zeta}} h(z,\zeta)$$

is the Bergman kernel, and the function l is defined by

$$l(z,\zeta) = \frac{1}{\pi} \frac{1}{(z-\zeta)^2} + \frac{2}{\pi} \frac{\partial^2}{\partial z \partial\zeta} h(z,\zeta) .$$

It is not dificult to see that the singularity of h disappears when this function is differentiated as in the equation relating it to the kernel K. Also, by a theorem in [14], l is actually regular in D_1.

Let h_0, K_0 and l_0 denote the corresponding quantities when $D_1 = D$ is the unit disc. Since

$$h_0(z,\zeta) = \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right| ,$$

one finds that

$$K_0(z,\bar\zeta) = \frac{1}{\pi}(1 - z\bar\zeta)^{-2}$$

and

$$l_0(z,\zeta) = 0\,.$$

Thus on D, (3.3.1) gives the criterion of Nehari, namely that

$$|\{\psi, z\}| \leq \frac{6}{(1-|z|^2)^2}$$

is necessary for the univalence of ψ.

Let now F be a conformal diffeomorphism of D_1 onto D. Then,

$$h(z,\zeta) = h_0(F(z), F(\zeta))\,,$$

and in differentiating this equation one obtains

$$K(z,\bar\zeta) = K_0(F(z), \overline{F(\zeta)})F'(z)\overline{F'(\zeta)}$$

and

$$l(z,\zeta) = -\frac{1}{\pi}\left(\frac{F'(z)F'(\zeta)}{(F(z)-F(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right)\,.$$

Hence, $(K(z,\bar z))^{1/2}|dz|$ is the Poincaré metric on D_1 and $l(z,z) = -\frac{1}{\pi}\{\psi, z\}$. We now state

Theorem 3.3.1 *With the notation as before, if*

$$|\frac{1}{\pi}U_\psi(z,z) + l(z,z)| \leq \frac{1}{3}K(z,\bar z) \tag{3.3.2}$$

then ψ is univalent.

Proof: We will derive (3.3.2) from Theorem 3.1.2 applied to D_1 with its Poincaré metric $g = K(z,\bar z)g_0$. Let $\varphi = \log|\psi'|$, then

$$\psi^*(g_0) = e^{2\varphi}g_0 = e^{2\varphi}K^{-1}g$$

and thus

$$S_g(\varphi) = B_g(\varphi - \frac{1}{2}\log K) = B_g(-\frac{1}{2}\log K) + B_{g_0}(\varphi) = B_{g_0}(\varphi) - B_{g_0}(\frac{1}{2}\log K)\,.$$

By the conformal invariance of K,

$$\frac{1}{2}\log K = \frac{1}{2}\log(K_0 \circ F) + \sigma$$

where $\sigma = \log|F'|$. Hence,

$$\begin{aligned} B_{g_0}(\frac{1}{2}(\log K)) &= B_{g_0}(\frac{1}{2}\log(K_0 \circ F) + \sigma) \\ &= B_{g_0}(\sigma) + B_{e^{2\sigma}g_0}(\frac{1}{2}\log(K_0 \circ F)) \\ &= B_{g_0}(\sigma) + F^*(B_{g_0}(\frac{1}{2}\log K_0)) \\ &= B_{g_0}(\sigma) \end{aligned}$$

because $B_{g_0}(\frac{1}{2}(\log K_0)) = 0$. Therefore

$$S_g(\psi) = B_{g_0}(\varphi) - B_{g_0}(\sigma)\,,$$

and Theorem 3.1.2 becomes

$$||S_g(\psi)|| \le 2\pi$$

or

$$|\{\psi, z\} - \{F, z\}| \le 2\pi K(z, \bar{z})\,,$$

which gives (3.3.2).

As mentioned in the introduction, (3.3.1) holds also as a necessary condition for univalence on multiply connected domains. It was natural to seek a corresponding sufficient condition on such domains by using Theorem 3.1.2 probably with the Bergman metric $(K(z,\bar{z}))^{1/2}|dz|$. This metric is complete and has curvature ≤ -4 (see [48]). After spending some time with the pertinent calculations, we became aware of the following instance of the Theorem 3.1.2 :

Theorem 3.3.2 *Let (M, g) be a Riemannian manifold of dimension $n \ge 2$ and ψ : $(M, g) \to (S^n, g_1)$ a conformal local diffeomorphism. Suppose that the scalar curvature of M is bounded above by $n(n-1)K$ for some $K \in R$, and that any two points in M can be joined by a geodesic. If*

$$||S_g(\psi)|| \le -\frac{1}{2}K$$

then M is simply-connected.

Proof: Let $(\tilde{M}, \tilde{g})$ be the universal cover of M with the metric $\tilde{g} = \pi^*(g)$, where $\pi : \tilde{M} \to M$ is the covering map. Then the lift $\tilde{\psi} = \psi \circ \pi : (\tilde{M}, \tilde{g}) \to (S^n, g_1)$ is a conformal local diffeomorphism and satisfies

$$||S_{\tilde{g}}(\tilde{\psi})|| \leq -\frac{1}{2}\tilde{K}\,,$$

where $n(n-1)\tilde{K} = n(n-1)K$ is an upper bound for the scalar curvature of $\tilde{g}$. Therefore by the theorem of Osgood and Stowe, $\tilde{\psi}$ is univalent, which implies that π has degree 1. This proves the theorem.

Since the Bergman metric is complete, a theorem of injectivity derived from Theorem 3.1.2 in this metric would have to hold with $\delta = \infty$. But then by Theorem 3.3.2, a domain for which there exists an analytic function ψ satisfying this injectivity criterion would be forced to be simply-connected. In other words, such a criterion will be vacuous on planar domains of higher connectivity.

Chapter 4

Quasiconformal reflections

4.1 Quasiconformal reflections in the plane

It is not an uncommon phenomenon that a stronger form of a given injectivity criterion serves further as a criterion for the existence of quasiconformal extensions. Classical examples are [1], [27] and more recently, [22] and [9].

In this chapter we will use Epstein's techniques for constructing quasiconformal reflections in hyperbolic space to show that the theorem of Osgood and Stowe falls under the category mentioned above. At this point, we present a brief summary of the main ideas that we shall require from the work in [22]. We will omit proofs and refer the reader to the source [22] for more details.

Let Σ be a complete surface in hyperbolic 3-space. We will use the ball model B^3 for this space. A main theorem in [22] asserts that if the principal curvatures k_1, k_2 of Σ satisfy

$$|k_i| < 1\,,$$

then Σ is properly embedded and diffeomorphic to a disc. Furthermore, the asymptotic boundary $\partial_\infty \Sigma$ is a Jordan curve on the 2-sphere S^2.

The following is the basic idea used to construct reflections in S^2 associated to such surfaces Σ. Let N be a unit normal to Σ. Given $p \in \Sigma$, one can follow the geodesic through p normal to Σ in both directions for infinite time. This defines then the forward and backward Gauss images $G_+(p)$ and $G_-(p)$ as the asymptotic limits on

S^2. Another important result of Epstein in [22] is that under the present assumptions on Σ, i.e., completeness and the bounds on the principal curvatures, the maps G_+ and G_- so defined are in fact diffeomorphisms onto open and disjoint subsets Ω_+ and Ω_- contained in S^2. In addition, $\partial\Omega_+ = \partial\Omega_- = \partial_\infty\Sigma$ and

$$S^2 = \Omega_+ \cup \Omega_- \cup \partial_\infty\Sigma\,.$$

One can therefore define the reflection

$$\Lambda = G_- \circ G_+^{-1} : \Omega_+ \to \Omega_-\,,$$

which fixes pointwise the curve $\partial_\infty\Sigma$.

By analyzing the behavior of the principal curvatures of the surfaces Σ_s which evolve from Σ under the parallel flow, Epstein obtains the following relevant formula relating the quasiconformal distortion K of Λ to the principal curvatures of Σ:

$$K = max\left\{ \left|\frac{1-k_1}{1+k_1}\right|^{\frac{1}{2}} \left|\frac{1+k_2}{1-k_2}\right|^{\frac{1}{2}} ; \left|\frac{1+k_1}{1-k_1}\right|^{\frac{1}{2}} \left|\frac{1-k_2}{1+k_2}\right|^{\frac{1}{2}} \right\}.$$

Here K is to be evaluated at the point $\theta = G_+(p) \in \Omega_+$ while the right-hand side is at the point p. From this one concludes immediately that if $|k_i| \leq t < 1$, then the induced reflection Λ is actually quasiconformal on the sphere and so $\partial_\infty\Sigma$ is a quasicircle.

As Epstein shows in [22], the surface Σ can be recovered as the envelope of the family of its tangent horospheres. Such a horosphere will be denoted by $H(\theta, \rho(\theta))$, where $\theta \in \Omega_+$ is the point of contact with S^2 of the given horosphere tangent to Σ. The horospheric radius $\rho(\theta)$ is the hyperbolic distance from a fixed origin $\vartheta \in B^3$ to the horosphere $H(\theta, \rho(\theta))$, i.e.,

$$|\rho| = inf\{\, d(\vartheta, q) :\ q \in H(\theta, \rho)\}\,.$$

The convention on the sign of ρ is that it be positive if ϑ lies outside the horosphere and negative otherwise. Epstein also shows that the bounds $|k_i| < 1$ on the principal curvatures of Σ guarantee that a point of tangency between a horosphere and Σ is unique. In summary, given a point $\theta \in \Omega_+$, there exists a unique real number $\rho(\theta)$

such that the horosphere $H(\theta, \rho(\theta))$ is tangent to Σ. Results similar to these, that we shall require for the main theorem in this chapter, are true in higher dimensional hyperbolic space.

As a preliminary case, we consider on the unit disc D metrics of the form $g = e^{2\sigma}(1-|z|^2)^{-2}g_0$ that satisfy

$$\text{(I) } 1 + (1-|z|^2)^2\sigma_{z\bar{z}} > 0$$

and

$$\text{(II) } |\sigma_z|(1-|z|^2) \leq t|z|$$

for some $0 \leq t < 1$.

These kind of metrics were used by Epstein in [22], and we shall show that his main theorem there can be regarded as a strong version of Theorem 3.1.2. Let $k(g)$ be the Gauss curvature of the metric g, i.e.,

$$k(g) = -4e^{-2\sigma}(1 + (1-|z|^2)^2\sigma_{z\bar{z}}),$$

which by (I) is strictly negative. Then

Theorem 4.1.1 *(Epstein) Let ψ be analytic and locally univalent in D. If*

$$||S_g(\psi)|| \leq -\frac{1}{2}tk(g) \tag{4.1.1}$$

then ψ is univalent and admits a $\frac{1+t}{1-t}$-quasiconformal extension to entire plane.

Proof: We remark that (4.1.1) is a pointwise inequality. Because of (II), the metric g is complete and therefore any two points in D can be joined by a geodesic in the metric g. Hence the univalence of ψ follows from the criterion of Osgood and Stowe.

The proof is basically the same as in Theorem 3.2.1. In fact, a calculation as in that theorem shows that

$$4\frac{||S_g(\psi)||}{k} = \left|\frac{(1-|z|^2)^2(\sigma_{zz} - \sigma_z^2 - \frac{1}{2}\{\psi, z\}) - 2\bar{z}(1-|z|^2)\sigma_z}{1 + (1-|z|^2)^2\sigma_{z\bar{z}}}\right|.$$

We outline the argument in [22]. Assume first that ψ is regular in $\bar{D}$. A surface Σ is constructed as the envelope of the family of horospheres $\{H(\theta, \rho(\theta)) : \ \theta \in \psi(D)\}$, where the support function ρ is determined by the equation

$$e^{2\rho} g_1 = \phi^*(g)$$

with $\phi = \psi^{-1}$. Most of the work relies on showing that Σ is complete and has principal curvatures $|k_i| < 1$. This will then guarantee that the reflection Λ is quasiconformal, which enables one to define the desired extension of ψ by appropriate conjugation.

The completeness of Σ follows essentially from the fact that $\rho \to \infty$ near $\partial\psi(D)$ (it is here that one needs ψ to be regular on $\bar{D}$). The principal curvatures of Σ are estimated by computing the Beltrami coefficient of Λ. In the upper half model H^3 of hyperbolic space with $\partial H^3 = C \cup \{\infty\}$, this coefficient is given by

$$\mu = \frac{f_{zz} - f_z^2}{f_{z\bar{z}}} \, ,$$

where the function f is defined by $f = \rho \circ S^{-1} - \log(1 + |x|^2)$ and $x = S(\theta)$ is the stereographic coordinate in the plane. Because the conformal factor $-\log(1 + |x|^2)$ arising from the spherical metric is Móbius with respect to g_0, and using the addition formula (1.2.4) it is not difficult to verify that

$$|\mu| = 4 \frac{||B_{g_1}(\rho)||_{g_2}}{|\tau|} \, ,$$

where τ is the curvature of the metric $g_2 = e^{2\rho} g_1$. Using (1.2.4) once more, one finally finds that

$$||S_g(\psi)|| = ||B_{g_1}(\rho)||_{g_2}$$

and thus by (4.1.1), $||\mu||_\infty \leq t$.

It is this last fact, namely that $|\mu|$ equals the g_2-norm of the euclidean Schwarzian tensor of ρ divided by the curvature of the metric g_2, what constitutes the essence of the corresponding theorem in arbitrary dimensions.

In the general case when ψ is not necessarily regular on $\bar{D}$, one has to look at a sequence of converging quasiconformal extensions of functions $\psi_n(z) = \psi(r_n z)$, $r_n \uparrow 1$.

There is a very geometric way in which the Schwarzian tensor of Osgood and Stowe comes up, and it is also in this context of surfaces in hyperbolic 3-space. Let ψ be a locally injective analytic map defined in the unit disc D. The best Möbius approximation to ψ at $z_0 \in D$ is defined to be the unique Möbius transformation F such that $F(z_0) = \psi(z_0)$, $F'(z_0) = \psi'(z_0)$ and $F''(z_0) = \psi''(z_0)$. In other words, $F^{-1} \circ \psi$ has the same 2-jet as the identity at z_0 and furthermore, it is not difficult to verify that $(F^{-1} \circ \psi)'''(z_0) = \{\psi, z_0\}$.

Our purpose is to recover the Schwarzian derivative of ψ in a slightly different way, as follows. For each $z \in D$, the aforementioned Möbius transformation F^z can be extended uniquely to a Möbius selfmap of the upper half space H^3. We will denote this extension again by F^z. Let $T(z)$ be the point on the unit sphere Σ_0 where a horosphere tangent at z is internally tangent to Σ_0.

As z varies through D, the point $R(z) = F^z(T(z))$ describes a surface Σ in H^3, and we will show

Theorem 4.1.2 *The pullback under R of the second fundamental form h of Σ in the hyperbolic metric equals the Schwarzian tensor of ψ in the euclidean metric.*

Proof: The proof is a rather long but hopefully amusing calculation. The extension of $F(\zeta) = \frac{a\zeta+b}{c\zeta+d}$ to an isometry of hyperbolic space is given by

$$F(\zeta + sj) = \frac{(a\zeta + b)\overline{(c\zeta + d)} + a\bar{c}s^2 + sj}{|c\zeta + d|^2 + |c|^2 s^2}\,, \tag{4.1.2}$$

where we have normalized so that $ad - bc = 1$ [11]. Also, j stands for the point (0,0,1). The explicit expression for $T(z)$ is

$$T(z) = \frac{2z}{1 + |z|^2} + \frac{1 - |z|^2}{1 + |z|^2} j\,. \tag{4.1.3}$$

We shall work at $z = 0$, and since the second fundamental form h is invariant under the isometries of H^3, we can assume that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi''(0) = 0$. Therefore F^0 is the identity and we will compute first $h(V, V)$ for any tangent vector V to Σ at j. At this point, the hyperbolic and euclidean metrics coincide and thus

$$h(V, V) = \langle \hat{\nabla}_V V, N \rangle\,, \tag{4.1.4}$$

where $\langle\ ,\ \rangle$ is the euclidean inner product, N the upward unit normal to Σ and $\hat{\nabla}$ the hyperbolic covariant derivative. Let $z_0 = x_0 + iy_0$ be fixed and let $z_t = T(tz_0)$. We consider

$$V = \frac{d}{dt}R(tz_0) = \frac{d}{dt}(F^{tz_0}(z_t)) = DF^{tz_0}(z_t)(z_t') + \frac{\partial F^{tz_0}}{\partial t}(z_t)\,, \tag{4.1.5}$$

where $DF^{tz_0}(z_t)$ is the differential of F^{tz_0} at z_t, and where $\frac{\partial F^{tz_0}}{\partial t}$ stands for the element of the Lie algebra of the group of isometries of H^3. We have

$$\frac{\partial F^z}{\partial t} = \frac{\partial F^z}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial F^z}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial t}\,. \tag{4.1.6}$$

Here, $\frac{\partial F^z}{\partial z}$ and $\frac{\partial F^z}{\partial \bar{z}}$ means differentiating the parameters a, b, c, d in (4.1.2). It is easy to see that a, b, c, d depend holomorphically on z, and using (4.1.3) we obtain at $t = 0$ (where $a = d = 1\ ,\ b = c = 0$)

$$V = 2z_0 + \overline{c' z_0}\,,\ c' = \frac{dc}{dz}(0)\,. \tag{4.1.7}$$

Hence the tangent plane to Σ at j is horizontal, and so $N = j$.

If we write the hyperbolic metric as $e^{2\varphi}\langle\ ,\ \rangle$, then

$$\hat{\nabla}_V V = \frac{dV}{dt} + 2\langle V, grad\ \varphi\rangle - \langle V, V\rangle grad\ \varphi\,, \tag{4.1.8}$$

with $grad\ \varphi$ euclidean. At j , $grad\ \varphi = -j$ and so

$$\hat{\nabla}_V V = \frac{dV}{dt} + \langle V, V\rangle j\,. \tag{4.1.9}$$

In differentiating (4.1.5) we conclude

$$\frac{dV}{dt} = DF^{tz_0}(z_t)(z_t'') + 2\frac{\partial(DF^{tz_0}(z_t))}{\partial t}(z_t') + \frac{\partial^2 F^{tz_0}}{\partial t^2}(z_t) + D^2F^{tz_0}(z_t', z_t')\,. \tag{4.1.10}$$

Since F^0 is the identity, at $t = 0$ we have $DF^{tz_0}(z_t)(z_t'') = z_t''$ and the last term on the right-hand side (the Hessian of F^{tz_0}) vanishes. Thus we are left to calculate $\frac{\partial^2 F^{tz_0}}{\partial t^2}(z_t)$ and $\frac{\partial(DF^{tz_0}(z_t))}{\partial t}(z_t')$ at $t = 0$. Most of the computations are tedious but straightforward, and we just point the important steps.

We use the chain rule to differentiate (4.1.6), and from (4.1.2) one finds that at $t = 0$,

$$\begin{aligned} \frac{\partial^2 F^{tz_0}}{\partial t^2}(z_t) &= z_0^2\{b'' - 2b'd' + (2d'^2 - d'')j\} + \bar{z}_0^2\{\bar{c''} - 2\bar{c'}\bar{d'} + (2\bar{d'}^2 - \bar{d''})j\} \\ &\quad + |z_0|^2\{a'\bar{c'} - d'\bar{c'} + (|d'|^2 - |c'|^2)j\}\,. \end{aligned} \tag{4.1.11}$$

From (4.1.2), with $\zeta = x + iy$ and $F^z(\zeta + sj) = u + iv + wj$ we can compute the differential of F^z to find

$$DF^z = \begin{pmatrix} u_x & u_y & u_s \\ v_x & v_y & v_s \\ w_x & w_y & w_s \end{pmatrix}, \tag{4.1.12}$$

with the following expressions

$$\begin{aligned} u_x + iv_x &= (|c|^2 + |d|^2)^{-2}\{(|c|^2 + |d|^2)(a\bar{d} + b\bar{c}) - (a\bar{c} - b\bar{d})(c\bar{d} + \bar{c}d)\}\,, \\ u_y + iv_y &= (|c|^2 + |d|^2)^{-2}\{(|c|^2 + |d|^2)(a\bar{d} - b\bar{c}) - (a\bar{c} + b\bar{d})(c\bar{d} - \bar{c}d)\}\,, \\ u_s + iv_s &= 2(|c|^2 + |d|^2)^{-2}\bar{c}\bar{d}\,, \end{aligned} \tag{4.1.13}$$

and

$$\begin{aligned} w_x &= -(|c|^2 + |d|^2)^{-2}(c\bar{d} + \bar{c}d)\,, \\ w_y &= -i(|c|^2 + |d|^2)^{-2}(c\bar{d} - \bar{c}d)\,, \\ w_s &= (|c|^2 + |d|^2)^{-2}(|d|^2 - |c|^2)\,. \end{aligned} \tag{4.1.14}$$

We now differentiate the components of DF^z using the chain rule and, as before, the fact that a, b, c, d depend on z analytically. Setting $t = 0$ finally yields

$$\frac{\partial}{\partial t}DF^{tz_0}(z_t) = \begin{pmatrix} Re\{z_0(a' - d')\} & Re\{z_0(a' - d')\} & Re\{2\bar{z}_0\bar{c'}\} \\ Im\{z_0(a' - d')\} & Im\{z_0(a' - d')\} & Im\{2\bar{z}_0\bar{c'}\} \\ -Re\{2\bar{z}_0\bar{c'}\} & -Im\{2\bar{z}_0\bar{c'}\} & -Re\{2z_0d'\} \end{pmatrix}. \tag{4.1.15}$$

Before going back to equation (4.1.8), we will find the appropriate relations between $a', b'.c', d'$ and d'' at $t = 0$. By construction,

$$\psi(z) = \frac{az+b}{cz+d} \ , \ \ \psi'(z) = \frac{1}{(cz+d)^2}$$

and

$$\psi''(z) = \frac{-2c}{(cz+d)^3} \,. \tag{4.1.16}$$

(Remember that we have chosen $ad - bc = 1$.) Therefore $\psi'''(0) = -2c'(0)$ and because of our normalizations on ψ, this gives $\{\psi, 0\} = -2c'(0)$.

On the other hand, we can explicitly solve for c and d to get

$$4c^2 = (\frac{\psi''}{\psi'})^2 \ , \ \ cz + d = -2c(\frac{\psi'}{\psi''}) \,,$$

from which

$$d'(0) = 0 \ \text{ and } \ d''(0) = -c'(0) \,.$$

From $ad - bc = 1$ we get $a'd + ad' - b'c - bc' = 0$, and thus at $z = 0$, $a' = -d' = 0$. The rest of the unknown in equation (4.1.10), i.e., b', b'' and c'', actually do not matter since we are going to take inner product with j. Aside from this, we want to derive an interesting equality. From

$$\psi(z) = \frac{az+b}{cz+d} \,,$$

we get by differentiating

$$\psi'(z) = \frac{1}{(cz+d)^2} + \frac{(cz+d)(a'z+b') - (az+b)(c'z+d')}{(cz+d)^2} \,,$$

and since by construction

$$\psi'(z) = \frac{1}{(cz+d)^2} \,,$$

we conclude that

$$\psi(z) = \frac{az+b}{cz+d} = \frac{a'z+b'}{c'z+d'} \,,$$

that is, the best Möbius approximation to ψ at z and the Möbius transformation determined by $a'(z), b'(z), c'(z)$ and $d'(z)$ agree at z. Back to our calculations, using equations (4.1.8) through (4.1.12) and that $z_t'(0) = 2z_0$ and $z_t''(0) = -4|z_0|^2 j$, we can now write

$$\hat{\nabla}_V V = \left(-4|z_0|^2 - 4(\alpha x_0 + \beta y_0) - (|c'|^2|z_0|^2 - 2Re\{c' z_0^2\}) + |V|^2\right) j + v_{hor} \,.$$

Here $\langle v_{hor}, j\rangle = 0$ and $\alpha + i\beta = 2\bar{c'}\bar{z}_0$. Since $V = 2z_0 + \bar{c'}\bar{z}_0$ we have

$$|V|^2 = (4 + |c'|^2)|z_0|^2 + 4Re(c' z_0^2) \,.$$

Thus finally

$$\begin{aligned} h(V,V) &= \langle \hat{\nabla}_V V, j\rangle \\ &= 4(\alpha x_0 + \beta y_0) - 6Re(c' z_0^2) \\ &= 2\{c_1(x_0^2 - y_0^2) - 2c_2 x_0 y_0\} \end{aligned}$$

where

$$c_1 + ic_2 = c' = -\frac{1}{2}\{\psi, 0\} \,.$$

Therefore the Schwarzian tensor of ψ at 0 is given by

$$S(\psi) = -2\begin{pmatrix} c_1 & -c_2 \\ -c_2 & -c_1 \end{pmatrix} ,$$

and we conclude that

$$h(V,V) = S(\psi)(z_0, z_0) \ , \ V = R_*(z_0) \,.$$

Since both forms h and $S(\psi)$ are symmetric, this shows that

$$h(R_*(z_0), R_*(z_1)) = S(\psi)(z_0, z_1) \,,$$

that is

$$R^*(h) = S(\psi) \,.$$

This finishes the proof of the theorem.

The principal curvatures of Σ are the eigenvalues of the form h, i.e., the maximum and minimum of $h(V,V)$ for unit V in the hyperbolic metric. At j this metric equals the euclidean metric, and we thus have to find the extrema of $\frac{h(V,V)}{|V|^2}$. We have already seen that for $V = R_*(z)$,

$$\begin{aligned} |V|^2 &= (4+|c'|^2)|z|^2 + 4Re(c'z^2) \\ &= (4+|c'|^2)|z|^2 - 2S(\psi)(z,z)\,. \end{aligned}$$

Note that the forms $R^*(h)$ and $\langle\ ,\ \rangle$ share eigenspaces, and since the eigenvalues of $S(\psi)$ are $2|c'|^2$ and $-2|c'|^2$, that is, $|\{\psi,z\}|$ and $-|\{\psi,z\}|$, we conclude that the principal curvatures of Σ are bounded in absolute value by

$$\frac{2|c'|}{(2-|c'|)^2} = \frac{4|\{\psi,z\}|}{(4-|\{\psi,z\}|)^2}\,.$$

In fact, one can find these curvatures explicitly by using the method of Lagrange multipliers to find the extrema of

$$\frac{S(\psi)(z,z)}{(4+|c'|^2)|z|^2 - 2S(\psi)(z,z)}$$

for $|z| = 1$. They are then given by

$$k_1 = \frac{2|c'|}{(2-|c'|)^2} = -k_2\,.$$

Then, for example, the surface Σ is minimal in the euclidean metric if $k_1 + k_2 = 2$, that is , when

$$|c'|^2 = 4(2+\sqrt{3}) \text{ or } |c'|^2 = 4(2-\sqrt{3})\,.$$

In other words, this will happen if

$$|\{\psi,z\}| = 4(2+\sqrt{3})^{\frac{1}{2}} \text{ or } |\{\psi,z\}| = 4(2-\sqrt{3})^{\frac{1}{2}}\,.$$

We make a final remark on envelopes and horospheres. The unit sphere Σ_0 is the envelope of a family of horospheres tangent to the unit disc in the plane. The horospheric radius $\rho(z)$ of the horosphere H^z tangent to D at z can be computed to be

$$\rho(z) = \log\frac{1+|z|^2}{1-|z|^2}\,.$$

(This is just the hyperbolic distance from j to H^z.) One could expect the surface Σ to be the envelope of the horospheres $F^z(H^z)$, but in general, if the principal curvatures of Σ are in absolute value bigger than 1, then there is no guaranty that $F^z(H^z)$ will be tangent to Σ or tangent at a single point. In any case, one can derive the following formula: if we denote by $\rho(F(H^z))$ the horospheric radius of the image of H^z under a Möbius transformation F, then

$$\rho(F(H^z)) = \log \frac{1+|F(z)|^2}{1-|z|^2} - \log |F'(z)| \,.$$

If we let $F = F^z$, we conclude that

$$\begin{aligned} \rho(F^z(H^z)) &= \log \frac{1+|\psi(z)|^2}{1-|z|^2} - \log |\psi'(z)| \\ &= \log \frac{|\phi'(\zeta)|}{1-|\phi(\zeta)|^2} + \log(1+|\zeta|^2)\,, \end{aligned}$$

where $\zeta = \psi(z)$ and ϕ is the local inverse of ψ such that $\phi(\zeta) = z$. We thus notice that $\rho(F^z(H^z))$ coincides with the support function used by Epstein to obtain quasiconformal reflections in the case of dealing just with the Poincaré metric in D. This is another way of recovering the fact that the quasiconformal extension in the criterion

$$|\{\psi, z\}| \leq \frac{2t}{(1-|z|^2)^2}\ ,\ 0 \leq t < 1$$

is given by letting the best Möbius approximation to ψ at $z \in D$ act on the reflected point $-1/\bar{z}$.

4.2 Reflections in higher dimensions

We shall study now quasiconformal reflections in higher dimensions. The set-up is basically the same as before. The n-sphere is thought of as the ideal boundary of hyperbolic $n+1$-space, and given a support function ρ defined on a domain $\Omega \subset S^n$, we consider the envelope hypersurface Σ to the family of horospheres $\{H(\theta, \rho(\theta)) : \theta \in \Omega\}$. The reflection $\Lambda = G_+^{-1} \circ G_-$ across Σ is given by

$$\Lambda(\theta) = \frac{|d\rho|^2 - 1}{|d\rho|^2 + 1}\theta + \frac{2d\rho}{|d\rho|^2 + 1}\,, \tag{4.2.1}$$

where $d\rho$ stands for the spherical gradient of ρ, $|d\rho|$ for its length in the spherical metric. It is then easy to see that

$$d\rho = \frac{\Lambda - (\Lambda \cdot \theta)\theta}{1 - (\Lambda \cdot \theta)}\,, \tag{4.2.2}$$

where $\cdot$ is the euclidean inner product (points on S^n are considered as being in R^{n+1}). All this can be found in [24].

We want to express $d\rho$ in terms of the stereographic coordinate $x = S(\theta)$ and the reflection

$$w = S \circ \Lambda \circ S^{-1}\,.$$

Let X_i be the vector field on S^n defined by $S_*(X_i) = \partial_i$. Then

$$d\rho(X_i) = d\rho \cdot X_i = \frac{\Lambda \cdot X_i}{1 - (\Lambda \cdot \theta)}\,.$$

We now use the equations

$$\Lambda \circ S^{-1} = S^{-1} \circ w = (1 + |w|^2)^{-1}(2w_1, ..., 2w_n, |w|^2 - 1)$$

and

$$X_i = 2(1 + |x|^2)^{-2}(-2x_1x_i, ..., 1 + |x|^2 - 2x_i^2, ..., -2x_nx_i, 2x_i)$$

to obtain

$$d\rho(X_i) = \frac{2x_i}{1 + |x|^2} + 2\frac{w_i - x_i}{|w - x|^2}\,. \tag{4.2.3}$$

We thus define

$$f = \rho \circ S^{-1} - \log(1 + |x|^2)\,,$$

and so (4.2.3) yields

$$grad\ f = 2\frac{w - x}{|w - x|^2}$$

or

$$w = x + 2\frac{grad\ f}{|grad\ f|^2}\,, \tag{4.2.4}$$

and here *grad* f stands for the euclidean gradient of f. We want to express the distortion of w in terms of f. One way to define such a distortion is as follows (Ahlfors): let Dw be the differential of w, and look at the eigenvalues $\lambda_1 \geq \lambda_2 \geq$

$\cdots \lambda_n \geq 0$ of the (positive) symmetric matrix $(Dw)^t(Dw)$. The map w is said to be $K-$quasiconformal if $\lambda_1\lambda_n^{-1} \leq K^2$.

We therefore need to find upper and lower bounds for $|Dw(y)|^2$, where $y \in R^n$ is a unit tangent vector at the point where the differential Dw is being considered. From (4.2.4),

$$Dw = I + 2DJ(grad\ f) \circ H(f) \tag{4.2.5}$$

where $J(x) = \frac{x}{|x|^2}$ the inversion in R^n. Its differential at the point x is given by

$$DJ = |x|^{-4}(|x|^2 I - 2Q(x))\,;$$

here $Q(x)$ is the symmetric matrix with i,j-component $x_i x_j$. Note that $Q^2(x) = |x|^2 Q(x)$ and thus DJ is a conformal matrix such that $|DJ| = |x|^{-2}$. Also, $H(f)$ stands for the Hessian of f, and in (4.2.5) DJ is evaluated at $x = grad\ f$.

So we have

$$Dw(y) = y + 2DJ(grad\ f)(H(f)(y))\,.$$

We now compute

$$\begin{aligned}
|Dw(y)|^2 &= 1 + 4|grad\ f|^{-4}|H(f)(y)|^2 + 4\langle DJ(grad\ f)(H(f)(y)), y\rangle \\
&= 1 + 4|grad\ f|^{-4}|H(f)(y)|^2 + 4\langle H(f)(y), DJ(grad\ f)(y)\rangle\,.
\end{aligned}$$

The Schwarzian tensor of f with respect to the euclidean metric is defined so that the matrix $B(f)$ representing it, is given by

$$B(f) = H(f) - Q(grad\ f) - \alpha I\,,$$

where $\alpha = \frac{1}{n}(\Delta f - |grad\ f|^2)$. Thus

$$\langle H(f)(y), y\rangle = \langle B(f)(y), y\rangle + \langle Q(grad\ f)(y), y\rangle + \alpha$$

and

$$\begin{aligned}
|H(f)(y)|^2 &= |B(f)(y)|^2 + |Q(grad\ f)(y)|^2 + \alpha^2 + 2\langle B(f)(y), Q(grad\ f)(y)\rangle \\
&\quad +2\alpha\langle B(f)(y), y\rangle + 2\alpha\langle Q(grad\ f)(y), y\rangle
\end{aligned}$$

and

$$\begin{aligned}\langle H(f)(y), Q(grad\ f)(y)\rangle &= \langle B(f)(y), Q(grad\ f)(y)\rangle + |grad\ f|^2\langle Q(grad\ f)(y), y\rangle \\ &\quad +\alpha\langle Q(grad\ f)(y), y\rangle\,.\end{aligned}$$

With this we obtain

$$\begin{aligned}|Dw(y)|^2 &= 1 + 4|grad\ f|^{-4}|B(f)(y)|^2 + 4|grad\ f|^{-4}|Q(grad\ f)(y)|^2 \\ &\quad +4|grad\ f|^{-4}(2\alpha + |grad\ f|^2)\langle B(f)(y), y\rangle + 4\alpha|grad\ f|^{-2} \\ &\quad +4\alpha^2|grad\ f|^{-4} - 4|grad\ f|^{-2}\langle Q(grad\ f)(y), y\rangle \\ \\ &= (1 + 2\alpha|grad\ f|^{-2})^2 + 4|grad\ f|^{-4}(|B(f)(y)|^2 + \beta\langle B(f)(y), y\rangle) \\ &\quad +4|grad\ f|^{-4}(|Q(grad\ f)(y)|^2 - |grad\ f|^2\langle Q(grad\ f)(y), y\rangle)\,,\end{aligned}$$

where

$$\beta = 2\alpha + |grad\ f|^2 = \frac{2}{n}\Delta f + (\frac{n-2}{n})|grad\ f|^2\,.$$

Using the fact that

$$\begin{aligned}|Q(grad\ f)(y)|^2 &= \langle Q(grad\ f)(y), Q(grad\ f)(y)\rangle \\ &= \langle Q^2(grad\ f)(y), y\rangle \\ &= |grad\ f|^2\langle Q(grad\ f)(y), y\rangle\,,\end{aligned}$$

we finally get

Proposition 4.2.1 *With the notation as before, the differential of the reflection w satisfies*

$$|Dw(y)|^2 = 4|grad\ f|^{-4}|A(y)|^2$$

where A is the matrix given by

$$A = \frac{1}{2}\beta I + B(f)\,.$$

On the other hand, the scalar curvature of the metric $\hat{g} = e^{2f}g_0$ is given by

$$scal(\hat{g}) = -n(n-1)e^{-2f}\beta\,,$$

and the norm of the tensor $B_{g_0}(f)$ in the metric $\hat{g}$ is given by $e^{-2f}||B(f)||_{g_0}$. Therefore, if

$$||B_{g_0}(f)||_{\hat{g}} \leq \frac{t}{2}\frac{|scal(\hat{g})|}{n(n-1)}$$

for some $0 \leq t < 1$, then

$$(1-t)|\beta||grad\ f|^{-2} \leq |Dw(y)| \leq (1+t)|\beta||grad\ f|^{-2}\,. \tag{4.2.6}$$

Hence for $|\beta||grad\ f| \neq 0$, the reflection w will be K- quasiconformal with $K = \frac{1+t}{1-t}$.

We consider now a conformal local diffeomorphism $\psi : (M, g) \to (S^n, g_1)$. We have seen that if $M = S^n$, then all such mappings ψ are just Möbius transformations. On the other hand, a large class of nontrivial maps ψ of this kind arise as the developing maps of locally conformally flat manifolds [50]. If

$$||S_g(\psi)|| \leq \frac{2\pi^2}{\delta^2} - \frac{1}{2}\frac{scal(g)}{n(n-1)}\,,$$

then ψ is a global diffeomorphism and with $\phi = \psi^{-1}$, we can define the metric $g_2 = e^{2\rho}g_1 = \phi^*(g)$ on $\Omega = \psi(M)$. As in the 2-dimensional case, the idea is to analyze the reflection Λ determined by Ω and the support function ρ. Using conformal invariance, we state now the following version of Theorem 3.3.2:

Theorem 4.2.1 *Let $\Omega \subset S^n$ be a domain with a complete metric $g_2 = e^{2\rho}g_1$. If*

$$||B_{g_1}(\rho)||_{g_2} \leq -\frac{1}{2}\frac{scal(g_2)}{n(n-1)}$$

then Ω is simply-connected.

Remarks

(1) The last inequality implicitly says that $scal(g) \leq 0$.

(2) One does not quite require that g_2 be complete, but rather that any two points in Ω can be joined (in Ω) by a geodesic in the metric g_2.

(3) This theorem is sharp, as can be verified by taking in the plane the ring $R_1 < |z| < R_2$ with its Poincaré metric. This metric satisfies the inequality

$$||B_{g_1}(\rho)||_{g_2} \leq -(1+\epsilon)\frac{1}{2}\frac{scal(g_2)}{2}\,,$$

where $\epsilon = \left(\frac{\log(R_2/R_1)}{\pi}\right)^2$ can be made arbitrarily small.

Proof: Let $\tilde{\Omega}$ be the universal cover of Ω with covering map π and metric $\tilde{g} = \pi^*(g_2)$. We consider π as a conformal map from $(\tilde{\Omega}, \tilde{g})$ into (S^n, g_1). Then, under the hypothesis of the theorem, one has

$$||S_{\tilde{g}}(\pi)||_{\tilde{g}} \leq -\frac{1}{2}\frac{scal(\tilde{g})}{n(n-1)},$$

which by Theorem 3.1.2 implies the univalence of π and consequently, our theorem.

We go back to the reflection Λ. We have shown that when $scal(\hat{g})|grad\ f| \neq 0$, the reflection Λ across Σ has a distortion bounded by $\frac{1+t}{1-t}$, where

$$t = \frac{2n(n-1)}{scal(\hat{g})}||B_{g_0}(f)||_{\hat{g}}\,.$$

The metrics $\hat{g}$ and g_2 are isometric under the stereographic projection S, and furthermore we claim that

$$||B_{g_0}(f)||_{\hat{g}} = ||B_{g_1}(\rho)||_{g_2}\ .$$

This follows from the addition formula, as:

$$0 = B_{g_0}(f-f) = B_{g_0}(f) + B_{e^{2f}g_0}(-f)\,.$$

But

$$B_{e^{2f}g_0}(-f) = (S^{-1})^*(B_{e^{2\rho}g_1}(-\rho)) = -(S^{-1})^*(B_{g_1}(\rho))\ .$$

Hence

$$B_{g_0}(f) = (S^{-1})^*(B_{g_1}(\rho))$$

and thus our claim follows.

On the other hand, Epstein has shown that this distortion equals

$$\max_{i\neq j}\left|(\frac{1+k_i}{1-k_i})(\frac{1-k_j}{1+k_j})\right|,$$

where $k_1, ..., k_n$ are the principal curvatures at the corresponding point $p = G_+^{-1}(\theta)$ on Σ.

We want to conclude that $|k_i| < 1$ for all i. To that effect, we will need to impose that the metric g_2 be negatively curved. This will replace the apparently awkward condition $|\beta||grad\ f| \neq 0$. We shall show that the assumption on the curvatures of g_2 together with the inequality

$$||B_{g_1}(\rho)||_{g_2} \leq -\frac{t}{2}\frac{scal(g_2)}{n(n-1)}$$

for some $0 \leq t < 1$, imply the sought estimate of the principal curvatures of Σ.

Following the work of Epstein, let Σ_s be the forward parallel hypersurface to Σ at distance s. The hyperbolic metric g_s on Σ_s, suitably normalized, converges as $s \to \infty$ to the metric g_2 on Ω [22]. The normalized sectional curvatures tend to $(k_ik_j - 1)(1-k_i)^{-1}(1-k_j)^{-1}$, and therefore $(k_ik_j - 1)(1-k_i)(1-k_j) < 0$. Hence $k_i \neq 1$ for all i. On the other hand, the principal directions of Σ and Σ_s are mapped to each other under the parallel flow, which enables one to compute the differential of Λ. In particular, its determinant is given by

$$-\prod_{i=i}^{n}(\frac{1+k_i}{1-k_i}),$$

where $\frac{1+k_i}{1-k_i}$ is the eigenvalue of $d\Lambda$ corresponding to the principal direction i. Since Λ reverses orientation, we conclude that

$$\prod_{i=1}^{n}(1-k_i^2) \geq 0\,.$$

We claim that this inequality is strict. Indeed, if not, then since we have already excluded the case $k_i = 1$, we must have $k_i = -1$ for some i. Because $k_i \neq 1$, $d\Lambda$ does not have an infinite eigenvalue and therefore $|grad\ f| \neq 0$ in (4.2.6). Hence the distortion is finite and we see from Epstein's formula that $k_j = -1$ for some $j \neq i$. This contradicts the fact that $(k_ik_j - 1)(1-k_i)(1-k_j) < 0$. This proves the claim, which now implies that $\#\{i : |k_i| > 1\}$ is even. If this number is not zero, then say $|k_1|, |k_2| > 1$. But then $(k_1k_2 - 1)(1-k_1)(1-k_2) > 0$, again a contradiction. Thus, $|k_i| < 1$ for all i, and therefore all the sectional curvatures of Σ are negative.

We now prove

Lemma 4.2.1 *If $\rho(\theta_n) \to \infty$ for any sequence $\{\theta_n\}$ in Ω converging (in the spherical metric) to a point in $\partial\Omega$, then Σ is complete.*

Proof: Suppose $\gamma(t)$ is a unit speed curve in Σ defined on [0,1) which cannot be extended continuously to $t = 1$ in Σ. Then the curve $G_+(\gamma(t))$ in Ω will have to tend to $\partial\Omega$, hence $\rho \to \infty$ along it. But then by construction of Σ as the envelope of the horospheres $H(\theta, \rho(\theta))$, we will have $\gamma(t)$ of infinite length, a contradiction.

Remark The hypothesis of the lemma will hold if, for instance, g_2 is a complete metric on Ω.

With this we conclude

Theorem 4.2.2 *Let g_2 have negative curvature and assume that $\rho \to \infty$ near $\partial\Omega$. If for some $t \in [0, 1)$*

$$||B_{g_1}(\rho)||_{g_2} \leq -\frac{t}{2}\frac{scal(g_2)}{n(n-1)}$$

then Ω is diffeomorphic to R^n.

Proof: The hypersurface Σ is complete with principal curvatures $|k_i| < 1$, and therefore the forward Gauss map $G_+ : \Sigma \to \Omega$ is a diffeomorphism. Hence Σ is simply-connected and by the Cartan-Hadamard theorem, it is diffeomorphic to R^n. This proves the result.

We now state the main result in this section.

Theorem 4.2.3 *Let (M, g) be a complete Riemannian n-manifold of negative curvature, and let $\psi : (M, g) \to (S^n, g_1)$ be a conformal local diffeomorphism such that for some $t \in [0, 1)$*

$$||S_g(\psi)|| \leq -\frac{t}{2}\frac{scal(g)}{n(n-1)}\,.$$

Then ψ is univalent and M diffeomorphic to R^n. Furthermore, there exists a $\frac{1+t}{1-t}$-quasiconformal diffeomorphism Λ of S^n onto itself, which takes the topological hemisphere $\Omega = \psi(M)$ to $S^n/\bar{\Omega}$ and which fixes $\partial\Omega$.

Remark Since g is complete, so is the metric g_2, but the proof of the theorem only requires that ρ becomes infinite near $\partial\Omega$.

Proof: The univalence of ψ follows from Theorem 3.1.2 and Theorem 3.3.2 implies that M is simply-connected. Hence, by the Cartan-Hadamard theorem, M is diffeomorphic to R^n. The stated inequality on $S_g(\psi)$ translates to

$$||B_{g_1}(\rho)||_{g_2} \leq -\frac{t}{2}\frac{scal(g_2)}{n(n-1)}\,,$$

and the remaining conclusions follow from the previous considerations on Λ.

Finally, we use the language of conformal geometry to give the following analytic characterization of quasidiscs in the plane.

Theorem 4.2.4 *Let $\Omega \subset R^2$ be a domain and let $g = e^{2f}g_0$ be a complete metric of negative Gaussian curvature $k(g)$. If for some $t \in [0,1)$*

$$||B_{g_0}(f)||_g \leq -\frac{t}{2}k(g) \tag{4.2.7}$$

then Ω is a quasidisc.

Remarks
(1) By Theorem 4.2.1, if the last inequality holds for $t = 1$, then one can only conclude that Ω is simply-connected.
(2) As in Theorem 4.2.1, one does not relly need g to be complete, but in this case the slightly weaker condition that the conformal factor $f \to \infty$ near $\partial\Omega$, and that any two points in Ω can be joined by a geodesic in the metric g.

Proof: This theorem is implicit in the work of Epstein in [22], but never stated in this intrinsic form. The function f, considered as a support function on Ω, determines a complete surface Σ in hyperbolic 3-space. By (4.2.7), which translates to

$$|f_{zz} - f_z^2| \leq t f_{z\bar{z}}\,,$$

and the fact that $k(g) < 0$, one sees from the argument presented from [22] that the principal of Σ are bounded in absolute value by 1. Then $\partial\Omega$ is the fixed point set of the associated quasiconformal reflection, which implies the result.

Chapter 5

Applications

5.1 Injectivity criterion for conformal immersions

In this section we will derive a sufficient condition for the conformal immersion of a Riemannian manifold into euclidean space to be an embedding. This will be done along the lines of the proof of the theorem of Osgood and Stowe, the main additional element being the study of the restriction to an n-dimensional submanifold of R^m of positive solutions to $Hess(u) = \frac{\Delta u}{m} g_0$. To that extent, we consider first the general case. Let M be an m-dimensional Riemannian manifold with metric g, and S an n-dimensional submanifold of M. Let ∇ denote the Levi-Cevita connection in M and $\hat{\nabla}$ the one for S in the induced metric. Then for X, Y tangent to S one has

$$\nabla_X Y = \hat{\nabla}_X Y + s(X, Y)\,,$$

where s is the second fundamental form of S. Let u be a positive function on M and let $v = -\log u$. Then,

$$Hess(u) - \frac{\Delta u}{m} g = -u B_g(v)\,.$$

We will denote by $\hat{}$ the corresponding quantities on S. At a point $x \in S$, $grad\ u$ decomposes uniquely in tangent and normal directions to S, as

$$grad\ u = T_u + N_u\,,$$

with similar notation for v.

Finally, let H be the mean curvature vector of S, i.e.,

$$H = \sum_{i=1}^{n} s(X_i, X_i)\,,$$

where $\{X_1, ..., X_n\}$ is an orthonormal basis for T_xS.

Lemma 5.1.1 *With the notation as above,*

$$uB_g(v) - u\hat{B}_g(v) = g(N_u, s - \frac{H}{n}g) + \frac{u}{n}\left(\sum_{i=1}^{n} B_g(v)(X_i, X_i)\right) g\,.$$

This should be understood as a tensor equality on S.

Proof: We compute first $H\hat{e}ss(u)$. By definition,

$$\begin{aligned} \widehat{Hess}(u)(X,Y) &= g(\hat{\nabla}_X T_u, Y) = g(\nabla_X T_u - s(X, T_u), Y) \\ &= g(\nabla_X T_u, Y) = g(\nabla_X grad\ u - \nabla_X N_u, Y) \\ &= Hess(u)(X,Y) + g(N_u, \nabla_X Y) \\ &= Hess(u)(X,Y) + g(N_u, s(X,Y))\,, \end{aligned}$$

that is,

$$\widehat{Hess}(u) = Hess(u) + g(N_u, s)\,.$$

Now,

$$\begin{aligned} -u\hat{B}_g(v) &= \widehat{Hess}(u) - \frac{\hat{\Delta} u}{n} g \\ &= Hess(u) + g(N_u, s) \\ &\quad -\frac{1}{n}\sum_{i=1}^{n} Hess(u)(X_i, X_i) - \frac{1}{n} g(N_u, H) g\,, \end{aligned}$$

and since

$$-uB_g(v) = Hess(u) - \frac{\Delta u}{m} g\,,$$

we get

$$\begin{aligned} -u\hat{B}_g(v) &= Hess(u) - \frac{1}{n}\left(-\sum_{i=1}^{n} uB_g(v)(X_i, X_i) + \frac{n}{m}\Delta u\right) g \\ &\quad + g(N_u, s - \frac{H}{n} g)\,. \end{aligned}$$

This gives the result.

The important application of this lemma will be when $B_g(v) = 0$, in which case we have

$$\hat{B}_g(v) = g(N_v, s - \frac{H}{n}g)\,.$$

This shows, for example, that $\hat{B}_g(v)$ also vanishes if S is totally umbilic.

Let now ψ be a conformal immersion of an n-dimensional manifold (M, g) into (R^m, g_0), with $\psi^*(g_0) = e^{2\varphi}g$. Let x and y in M be two given points joined by a geodesic γ of length $< \delta$. Let T be the unit tangent vector to γ. We want to establish a criterion under which $\psi(x) \neq \psi(y)$. Following the proof in [42] we consider the function

$$w = e^{-\varphi}(u \circ \psi)\,,$$

where

$$u(p) = |p - \psi(x)|^2$$

is the square of the euclidean distance to the point $\psi(x)$. The important features of u are: $u(p) > 0$ unless $p = \psi(x)$, and $Hess(u) = \frac{\Delta u}{m} g_0$, i.e., $B_{g_0}(-\log u) = 0$. Hence the function w vanishes at x, and in order to guarantee that it does not vanish at y, we will require w to satisfy a certain differential inequality along γ. Differentiation along this curve will be denoted by $'$. Then

$$w'' = Hess(w)(T, T) = -wB_g(-\log w)(T, T) + \frac{1}{n}\Delta w\,, \tag{5.1.1}$$

and

$$\begin{aligned} B_g(-\log w) &= B_g(\varphi - \log(u \circ \psi)) \\ &= B_g(\varphi) + B_{e^{2\varphi}g}(-\log(u \circ \psi)) \\ &= B_g(\varphi) + \psi^*(\hat{B}_{g_0}(-\log u)) \\ &= B_g(\varphi) + \psi^*(\hat{B}_{g_0}(v))\,. \end{aligned} \tag{5.1.2}$$

Here, $\hat{}$ denotes local computations on $\psi(M)$ with the induced metric. Let k and $\hat{k}$ be $(n(n-1))^{-1}$ times the scalar curvatures of g and $w^{-2}g$ respectively. Then

$$\hat{k} = w^2 \left(k + \frac{2}{n}\frac{\Delta w}{w} - \frac{|grad\ w|^2}{w^2} \right)\,, \tag{5.1.3}$$

hence

$$\frac{1}{n}\Delta w = \frac{w}{2}\left(-k + w^{-2}\hat{k} + \frac{|grad\ w|^2}{w^2}\right). \tag{5.1.4}$$

The inequality that we shall require is

$$w'' \geq -\frac{2\pi^2}{\delta^2}w + \frac{1}{2}\frac{(w')^2}{w}, \tag{5.1.5}$$

which can be written as

$$(w^{\frac{1}{2}})'' \geq -\frac{\pi^2}{\delta^2}(w^{\frac{1}{2}}). \tag{5.1.6}$$

If this is satisfied, a standard Sturm comparison theorem will then guarantee that w does not vanish at y. Using equations (5.1.1), (5.1.2) and (5.1.4), we can rewrite (5.1.5) as

$$\begin{aligned} -\frac{2\pi^2}{\delta^2}w + \frac{1}{2}\frac{(w')^2}{w} \quad &\leq \quad -wB_g(\varphi)(T,T) - w\hat{B}_{g_0}(\psi_*(T),\psi_*(T)) \\ &\quad + \frac{w}{2}\left(-k + w^{-2}\hat{k} + \frac{|grad\ w|^2}{w^2}\right). \end{aligned}$$

This last inequality will hold if

$$B_g(\varphi)(T,T) + B_{g_0}(v)(\psi_*(T),\psi_*(T)) \leq \frac{2\pi^2}{\delta^2} + \frac{1}{2}(w^{-2}\hat{k} - k). \tag{5.1.7}$$

We already have a more explicit expression for $\hat{B}_{g_0}(v)(\psi_*(T),\psi_*(T))$ as in the remark after the lemma, and now we want to derive another way of writing $w^{-2}\hat{k}$. The metric $w^{-2}g$ is (locally) isometric to $u^{-2}g_0$ restricted to $\psi(M)$. We point out that this last metric has a geometric interpretation: indeed, $u^{-2}g_0 = F^*(g_0)$, where F is the Möbius inversion in R^m given by

$$F(p) = \frac{p - \psi(x)}{|p - \psi(x)|^2}.$$

Hence, at a point $z \in M$, $\hat{k}$ equals $(n(n-1))^{-1}$ times the scalar curvature of $F(\psi(M))$ at $F(\psi(z))$ in the induced metric. Let k_1 be $(n(n-1))^{-1}$ times the scalar curvature of $\psi(M)$ in the induced mertic. Then

$$\hat{k} = u^2\left(k_1 + \frac{2}{n}\frac{\hat{\Delta}u}{u} - \frac{|T_u|^2}{u^2}\right),$$

where the left-hand side is at z and the right-hand side at $\psi(z)$. As before, $\hat{\Delta}$ will denote the Laplacian on $\psi(M)$.

As shown before,

$$\begin{aligned} \widehat{Hess}(u) &= Hess(u) + g_0(N_u, s) \\ &= \frac{\Delta u}{m} g_0 + g_0(N_u, s) \\ &= 2g_0 + g_0(N_u, s) \end{aligned}$$

since $\Delta u = 2m$. Thus

$$\hat{\Delta} u = 2u + g_0(N_u, H)\,.$$

On the other hand, $grad\ u = T_u + N_u$, therefore

$$|Tu|^2 = |grad\ u|^2 - |N_u|^2 = 4u - |N_u|^2\,.$$

So we obtain

$$\hat{k} = u^2\left(k_1 + \frac{4}{u} + \frac{2}{u} g_0(N_u, \frac{H}{n}) - \frac{4u - |N_u|^2}{u^2}\right)\,,$$

or equivalently

$$\hat{k} = u^2\left(k_1 + \frac{2}{u} g_0(N_u, \frac{H}{n}) + \frac{|N_u|^2}{u^2}\right)\,. \tag{5.1.8}$$

Notice that the term $u^{-1}|N_u|$ causes no trouble, since it remains bounded near $\psi(x)$.

Theorem 5.1.1 *With the notation as above, if along the geodesic γ*

$$B_g(\varphi)(T,T) + e^{2\varphi} g_0(N_v, s(T_1, T_1)) \leq \frac{2\pi^2}{\delta^2} + \frac{1}{2}(e^{2\varphi} k_1 - k + e^{2\varphi}|N_v|^2) \tag{5.1.9}$$

then $\psi(x) \neq \psi(y)$. Here, $T_1 = e^{-\varphi}\psi_*(T)$.

Proof: From the remark after the lemma, we have

$$\begin{aligned} \hat{B}_{g_0}(\psi_*(T), \psi_*(T)) &= g_0\left(N_v, s(\psi_*(T), \psi_*(T) - \frac{e^{2\varphi}}{n} H)\right) \\ &= e^{2\varphi} g_0\left(N_v, s(T_1, T_1) - \frac{1}{n} H\right)\,, \end{aligned}$$

and after this, (5.1.9) follows from equations (5.1.7) and (5.1.8).

An important case in which (5.1.9) simplifies, is when the immersion is isometric. Then φ vanishes identically and $k_1 = k$, which leads to

Theorem 5.1.2 *With the notation as before, if the immersion ψ is isometric and if along the geodesic γ*

$$g_0(N_v, s(T_1, T_1)) \leq \frac{2\pi^2}{\delta^2} + \frac{1}{2}|N_v|^2 \tag{5.1.10}$$

then $\psi(x) \neq \psi(y)$.

At this point we want to present two sharp applications of these theorems. First, we consider the inverse of the stereographic projection, $\psi : R^n \to S^n \subset R^{n+1}$. Here, $\varphi = -\log(1 + |x|^2)$ for $x \in R^n$. This embedding is Möbius, that is, $B_{g_0}(\varphi) = 0$. Also, $k = 0$ and $k_1 = 1$. Since S^n is totally umbilic,

$$g_0(N_v, s(N_1, N_1)) \leq |N_v|$$

and we see that by letting $\delta = \infty$, equation (5.1.9) will be satisfied for any two points in R^n. In other words, a Möbius immersion of R^n into euclidean space as a totally umbilic submanifold of constant scalar curvature is necessarily an embedding.

As a second example, we look at the isometric immersion of R^2 as a cylinder of radius r in R^3. We imagine the y-axis in R^2 as staying fixed, while the x-direction is rolled over to form the cylinder. In the y-direction, (5.1.10) is satisfied with $\delta = \infty$, whereas in the x-direction, we shall show that (5.1.10) will hold as long as $\delta \leq 2\pi r$. Indeed, it suffices to consider the case when $u(p) = |p|^2$ in R^3. Let N be the inward normal to the cylinder. Then

$$s(T_1, T_1) = \frac{N}{r},$$

and a simple calculation yields

$$N_v = -\frac{N_u}{u} = \frac{N}{r}.$$

Thus

$$\frac{1}{r^2} = g_0(N_v, s(T_1, T_1)) \leq \frac{2\pi^2}{\delta^2} + \frac{1}{2}|N_v|^2 = \frac{2\pi^2}{\delta^2} + \frac{1}{2r^2}$$

only if $\delta \leq 2\pi r$.

5.2 The complex analytic case

In this section, we shall discuss Theorem 5.1.1 in the complex analytic case. This will mainly involve a few calculations. As before, let D be the open unit disc in the plane and let $\psi : D \to C^n$ be holomorphic. Then ψ is conformal and $\psi^*(g_0) = e^{2\varphi} g_0$ with

$$\varphi = \frac{1}{2}\log(|\psi_1'|^2 + \cdots + |\psi_n'|^2)\,,$$

where $\psi = (\psi_1, ..., \psi_n)$ and $'$ denotes differentiation with respect to $z \in D$.

We write (5.1.9) as

$$B_{g_0}(\varphi)(T,T) + e^{2\varphi} g_0 \left(N_v, s(T_1,T_1) - \frac{1}{2}N_v\right) \leq \frac{2\pi^2}{\delta^2} + \frac{1}{2}(e^{2\varphi}k_1 - k)\,,$$

and here, $\delta = 2, k = 0$ and $e^{2\varphi}k_1 = -\Delta\varphi$. We will now compute $\Delta\varphi$ and the norm $||B_{g_0}(\varphi)||_{g_0}$. Using

$$\Delta\varphi = 4\varphi_{z\bar{z}}$$

and the holomorphicity of ψ, one finds that

$$\Delta\varphi = 2e^{-4\varphi}\sum_{i<j}|\psi_i'\psi_j'' - \psi_i''\psi_j'|^2\,.$$

On the other hand, it is easy to verify that

$$||B_{g_0}(\varphi)||_{g_0} = 2|\varphi_{zz} - \varphi_z^2|\,,$$

and after a computation one arrives at

$$\varphi_{zz} - \varphi_z^2 = \frac{1}{4}e^{-4\varphi}\left\{2(\sum_i|\psi_i'|^2)(\sum_i \psi_i'\bar{\psi}_i''') - 3(\sum_i \psi_i''\bar{\psi}_i')^2\right\}\,.$$

If λ is a pointwise upper bound for the second fundamental form s of the surface $\psi(D)$, that is, $|s(X,Y)| \leq \lambda$ for all unit tangent vectors X, Y at a given point $\psi(z)$, then we can rewrite equation (5.1.9) to obtain

Theorem 5.2.1 *If*

$$\frac{1}{2}e^{-4\varphi}\left|2(\sum_i|\psi_i'|^2)(\sum_i \psi_i'''\bar{\psi}_i') - 3(\sum_i \psi_i''\bar{\psi}_i')^2\right|$$
$$+e^{-4\varphi}\sum_{i<j}|\psi_i'\psi_j'' - \psi_i''\psi_j'| \;\leq\; \frac{\pi^2}{2} + \frac{1}{2}e^{2\varphi}(|N_v|^2 - 2\lambda|N_v|)$$

then ψ is univalent.

Finally, we mention that one can use our theorem in a similar way as a univalence criterion for C^m-valued holomorphic maps defined on domains in C^n or for that matter, on any n-dimensional complex manifold M. In fact, given $x, y \in M$ and assuming that they can be joined in M by a complex line, one only needs to consider the restriction of ψ to such curve.

5.3 Nonpositively curved target manifold

There are two important features about the test function u in the proof of the theorem of Osgood and Stowe which gives their result a very neat and concise statement. They are that the Hessian of u is diagonal and that the metric $u^{-2}g_0$ is flat. The purpose of this last section is to derive an analogous injectivity criterion when the target manifold $(N, \hat{g})$ is complete, simply-connected and nonpositively curved. As in the euclidean case, we will use as a test function u, the square of the distance to a given point. There is no hope that in this generality $u^{-2}\hat{g}$ will be flat, but one can nevertheless obtain relatively simple expressions relating the curvatures of $\hat{g}$ and $u^{-2}\hat{g}$. On the other hand, the size of $Hess(u) - \frac{\Delta u}{n}\hat{g}$ can be estimated by using comparison theorems. We will start with the pertinent computations to bound this last quantity.

Throughout this section, $(N, \hat{g})$ will be a complete, simply-connected n-dimensional Riemannian manifold with nonpositive sectional curvatures K, $-a^2 \le K \le 0$. Let u denote the square of the distance to a given point $p \in N$. Then u is smooth and everywhere positive, except at one point (where it vanishes). The main estimate we need is

Proposition 5.3.1

$$Hess(u)(X, X) - \frac{1}{n}\Delta u \ge \frac{2(n-1)}{n}\left(1 - (\frac{a}{2})(\frac{1+v^2}{v})u^{\frac{1}{2}}\right), \tag{5.3.1}$$

where X is a unit tangent vector and v is related to u by the equation

$$u^{\frac{1}{2}} = \frac{1}{a}\log\frac{1+v}{1-v}.$$

Proof: On euclidean space, the Hessian of the square of the distance to a given point equals twice the euclidean metric. Hence in our case, by the Hessian comparison theorem ([18], [28]),

$$Hess(u)(X,X) - \frac{1}{n}\Delta u \ \geq \ 2 - \frac{1}{n}\Delta s^2 \,, \tag{5.3.2}$$

where s is the distance function in hyperbolic space of constant curvature $-a^2$. To compute its (hyperbolic) Laplacian, we shall use the ball model B^n with metric $g = e^{2\varphi}g_0$, where

$$\varphi = \log \frac{A}{1-|x|^2} \ , \ A = \frac{2}{a} \, .$$

We have

$$Hess(s^2) = 2sHess(s) + 2ds \otimes ds \,,$$

and from the formula on how the covariant derivative changes under a conformal change of metric, we find that

$$Hess(s) = Hess_0(s) - d\varphi \otimes ds - ds \otimes d\varphi + g_0(grad_0\ s, grad_0\ \varphi)g_0 \, .$$

The subindex refers to computations in the euclidean metric g_0.

We can assume that the base point of s is the origin, and so

$$s = \int_0^{|x|} e^{\varphi(t)}dt = \frac{1}{a}\log\frac{1+|x|}{1-|x|} \, .$$

The vectors $grad_0 s$ and $grad_0 \varphi$ are parallel, and an easy computation shows that

$$g_0(grad_0\ s, grad_0\ \varphi) = \frac{4}{a}\frac{|x|}{(1-|x|^2)^2} \, .$$

Therefore,

$$Hess(u) = 2sHess_0(s) - 2sds \otimes d\varphi - 2sd\varphi \otimes ds + \frac{8s}{a}\frac{|x|}{(1-|x|^2)^2}g_0 + 2ds \otimes ds \, .$$

We now take trace in the metric g to conclude

$$\Delta u = 2se^{-2\varphi}\Delta_0 s - 4se^{-2\varphi}|grad_0\ s||grad_0\ \varphi| + 2nas|x| + 2e^{-2\varphi}|grad_0\ s|^2 \,,$$

where the norms of the gradients are euclidean. The terms left to compute are the euclidean Laplacian of s and $|grad_0\ s|^2$. This last one is

$$|grad_0\ s|^2 = \frac{4|x|^2}{(1-|x|^2)^2}\,,$$

and another simple calculation yields

$$\Delta_0 s = \frac{2n}{(1-|x|^2)^2} + \frac{2(2-n)|x|^2}{(1-|x|^2)^2}\,.$$

With this we finally obtain after some simplifications

$$\Delta u = 2 + (n-1)as\frac{1+|x|^2}{|x|}\,.$$

Inserted in equation (5.3.2), this gives the proposition.

It is not difficult to see that the right-hand side of (5.3.1) is $O(u)$ near the point p and $O(u^{\frac{1}{2}})$ when u is large. Such a behavior will matter at the end.

Let now (M,g) be an n-dimensional manifold and $\psi : M \to N$ a conformal local diffeomorphism. As mentioned before, we shall derive a sufficient condition for the global univalence of ψ. Let $\psi^*(\hat{g}) = e^{2\varphi}g$ and set

$$w = e^{-\varphi}(u \circ \psi)\,.$$

Hence $\psi^*(u^{-2}\hat{g}) = w^{-2}g$, and we will need to relate to each other the scalar curvatures of the involved metrics. Let $k,\ \hat{k}$ and k' denote respectively $(n(n-1))^{-1}$ times the scalar curvature of $g,\ \hat{g}$ and $u^{-2}\hat{g}$ $(w^{-2}g)$. Then

$$\begin{aligned} k' &= u^2(\hat{k} + \frac{2}{n}\frac{\Delta_{\hat{g}}u}{u} - \frac{|grad_{\hat{g}}\ u|^2_{\hat{g}}}{u^2}\hat{g}) \\ &= w^2(k + \frac{2}{n}\frac{\Delta_g w}{w} - \frac{|grad_g\ w|^2_g}{w^2}g)\,, \end{aligned} \tag{5.3.3}$$

where when computing quantities on M and N, they are to be evaluated respectively at $x \in M$ and $\psi(x) \in N$. For once, we have used metric subindices, but we shall drop them in the subsequent, with the convention that metric dependent quantities on M

are computed in g and those in N, in $\hat{g}$. From (5.3.3) we get

$$\begin{aligned} \frac{1}{n}\frac{\Delta w}{w} &= \frac{1}{2}(\frac{|grad\ w|^2}{w^2} - k) \\ &+\frac{1}{2}\frac{u^2}{w^2}(\hat{k} + \frac{2}{n}\frac{\Delta u}{u} - \frac{|grad\ u|^2}{u^2}) . \end{aligned} \tag{5.3.4}$$

Now, $u^2 w^{-2} = e^{2\varphi}$ and furthermore, we claim that

$$\frac{2}{n}\frac{\Delta u}{u} \geq \frac{|grad\ u|^2}{u^2} .$$

Indeed, since N is nonpositively curved, by the comparison theorems we have

$$\Delta u \geq n .$$

On the other hand, $u = s^2$, hence

$$|grad\ u|^2 = |2s grad\ s|^2 = 4s^2 .$$

Therefore

$$\frac{|grad\ u|^2}{u^2} = \frac{4}{u}$$

and the claim follows. So,

$$\frac{1}{n}\frac{\Delta w}{w} \geq \frac{1}{2}\frac{|grad\ w|^2}{w^2} - \frac{1}{2}k + \frac{1}{2}e^{2\varphi}\hat{k} . \tag{5.3.5}$$

The procedure now is essentially the same as in section 5.1. Let $x, y \in M$ be joined by a geodesic γ of length $< \delta \leq \infty$. Let the base point p of the function u be $\psi(x)$. Then w vanishes at x and $\psi(x) \neq \psi(y)$ iff $w(y) \neq 0$. Therefore, an injectivity criterion can be formulated as a differential inequality of w on γ that will ensure the nonvanishing of w before time δ. By the addition formula,

$$\begin{aligned} B_g(-\log w) &= B_g(\varphi - \log(u \circ \psi)) \\ &= B_g(\varphi) + B_{e^{2\varphi}g}(-\log(u \circ \psi)) \\ &= B_g(\varphi) + \psi^*(B_{\hat{g}}(-\log u)) , \end{aligned}$$

and by (1.2.9),

$$Hess(u) - \frac{\Delta u}{n}\hat{g} = -uB_{\hat{g}}(-\log u) .$$

So we obtain

$$B_g(-\log w) = B_g(\varphi) - \frac{1}{u(\psi)}\psi^*(Hess(u) - \frac{\Delta u}{n}\hat{g})\,. \tag{5.3.6}$$

Let T be the unit tangent along γ oriented from x to y. Then

$$\begin{aligned} w'' &= Hess(w)(T,T) = \frac{\Delta w}{n} - wB_g(-\log w)(T,T) \\ &= \frac{\Delta w}{n} - wB_g(\varphi)(T,T) + \frac{w}{u(\psi)}\left(Hess(u)(\psi_*(T),\psi_*(T)) - e^{2\varphi}\frac{\Delta u}{n}\right) \\ &= \frac{\Delta w}{n} - wB_g(\varphi)(T,T) + e^{2\varphi}(Hess(u)(X,X) - \frac{\Delta u}{n})\,, \end{aligned}$$

where we have written $\psi_*(T)$ as $e^{\varphi}X$. As in section 5.1, the differential inequality to be satisfied is

$$w'' \geq -\frac{2\pi^2}{\delta^2}w + \frac{1}{2}\frac{(w')^2}{w}\,,$$

which can be written as

$$(w^{\frac{1}{2}})'' \geq -\frac{\pi^2}{\delta^2}w^{\frac{1}{2}}\,. \tag{5.3.7}$$

The function w satisfies $w(x) = 0$ and $w > 0$ near x, therefore if equation (5.3.7) holds, a Sturm comparison theorem will guarantee that w cannot vanish at y. Using the estimates (5.3.1) and (5.3.5), we conclude

Theorem 5.3.1 *With the notation as before, if along the geodesic γ*

$$B_g(\varphi)(T,T) - \frac{2(n-1)}{n}\frac{e^{2\varphi}}{u(\psi)}\left(1 - \frac{1}{a}\left(\frac{1+v^2}{v}\right)u^{\frac{1}{2}}\right)(\psi) \leq$$
$$\frac{2\pi^2}{\delta^2} - \frac{1}{2}k + \frac{1}{2}e^{2\varphi}\hat{k}(\psi) \tag{5.3.8}$$

then $\psi(x) \neq \psi(y)$.

We make a few concluding remarks. It is not difficult to verify that

$$\frac{1}{u(\psi)}\left(1 - \frac{1}{a}(\frac{1+v^2}{v})u^{\frac{1}{2}}\right)(\psi) \leq 0\,,$$

so in the left-hand side this term contributes as a positive quantity. Nevertheless, near x it is bounded and tends to 0 as $u(\psi) \to \infty$. The map ψ will be a global diffeomorphism if (5.3.8) holds for all pairs x, y as before.

Finally, one could put together the results of this section and of section 5.1 to derive an injectivity criterion for the conformal immersion of an arbitrary manifold M into a higher dimensional complete and simply-connected nonpositively curved manifold N. This is so, because by using the results in section 5.1 we can relate in the target N the function $u = dist^2(\ ,p)$ to its restriction to the (local) submanifold $\psi(M)$. The appropriate estimates for the corresponding operators on u can be derived from Proposition 5.3.1.

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